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6D supersymmetric nonlinear sigma-models in 4D, $\mathcal{N} = 1$ superspace

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Abstract: Using 4D, $\mathcal{N} = 1$ superfield techniques, a discussion of the 6D sigma-model possessing simple supersymmetry is given. Two such approaches are described. Foremost it is shown that the simplest and most transparent description arises by use of a doublet of chiral scalar superfields for each 6D hypermultiplet. A second description that is most directly related to projective superspace is also presented. The latter necessarily implies the use of one chiral superfield and one nonminimal scalar superfield for each 6D hypermultiplet. A separate study of models of this class, outside the context of projective superspace, is also undertaken.

Keywords: Sigma Models, Extended Supersymmetry, Field Theories in Higher Dimensions, Superspaces.

Contents

1. Introduction

The topic of six dimensional supersymmetrical sigma-models [1, 2] is curiously one that has hardly been explored in the literature. Certainly one possible explanation for this is the expectation that no fundamentally new features will emerge. For example, since by reduction to 4D they become $\mathcal{N} = 2$ models, the already extensive literature on the latter must surely constitute an indirect study of these models and has already illustrated all structures of the 6D theories. However, this raises questions that always occur when discussions of compactifications are present in supersymmetrical theories. Are there features of the compactified theories that only occur in the lower dimension? How are the features that only permitted in the 6D theory to be disentangled from those that are present only in the compactified theory? Moreover, with the topic of 'little strings' [3] having been discovered, one would also prefer a study of 6D nonlinear sigma-model theory in an effort to find whether there are features of the former that are encoded in the structure of the latter. Finally, these studies of 6D models in terms of 4D, $\mathcal{N} = 1$ (or more generally higher D) models [4]–[6] opens up an arena for the study of the corresponding realizations of superconformal symmetry, supergravity and perhaps most fascinating of all, superstring/M-theory.

In previous work [7] we have probed the structure of the 6D hypermultiplet as viewed by the tool of a formulation that only realizes the full 6D Lorentz group fully on-shell but

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permits the realization of 4D, $\mathcal{N} = 1$ supersymmetry off-shell. Thus, the present work naturally follows onto this previous set of investigations. A summary of this work follows.

In the second chapter, the formulation of this class of models in terms of pair of chiral multiplets (CC formulation) is given. It is shown how the condition of on-shell Lorentz invariance naturally leads to the condition that the geometry of the nonlinear sigma-models must be that of a hyper-Kähler manifold [8]. The determinant of the hyper-Kähler metric is equal to the square modulus of the determinant of the exterior derivative of a holomorphic one-form which is related, in our 4D, $\mathcal{N} = 1$ superspace fomulation, to the extra-dimensions. This condition results to be equivalent to the Monge-Ampère equation and implies Ricci flatness. The triplet of complex structures that possess a quaternionic algebra is identified and related to the exterior derivative of the holomorphic one-form. With a correct definition of how to obtain the 6D component fields from the 4D ones, the on-shell action is found to take the expected form: Kinetic energies for the spin-zero and spin-1/2 states together with a quartic fermionic interaction that involves the Riemann tensor for the manifold geometry.

In the third chapter, an exploration of the origin of such models arising from projective superspace [9]–[15], [7] is undertaken. 'Projectivized' superderivatives are defined in the usual manner. This is followed by a review of the polar formulation of hypermultiplets and the discussion of sigma-model actions that can be introduced for these. As an example of the general structure of these 6D sigma-models we consider the particular case of tangent bundles of Kähler manifolds. Although no explicit results are given for the $O(2n)$ 6D $\mathcal{N}$ = (1, 0) multiplets, it is noted that the extension to the 6D arena is possible.

In the fourth chapter, we analyze the very difficult problem of deriving the geometry that arises in the case of directly using the CNM (chiral/non-minimal) [16]–[19] formulation without the starting point of projective superspace. The starting point for this mimics the techniques used in chapter two but includes now the complication to allow *both* chiral and complex linear superfields [20, 21] (i.e. non-minimal scalar multiplets) *ab initio* in the analysis. It is noted that whenever the number of nonminimal multiplets is less than the number of chiral multiplet, a subsector of the theory must take the form given in chapter two. Full expressions for the bosonic terms in the action, prior to removal of auxiliary fields are given. Imposing 6D Lorentz invariance, imposes a condition on the generalized potential in the model that is very similar to that found in the pure CC case. However, no simple solution to the general case of this system are obtainable by our present methods. An explicit solution is presented in a special case where an explicit proof is obtained that CNM geometry is a hyper-Kähler one.

In the fifth chapter, a discussion of the duality between the 6D CC and CNM formulations is undertaken. Once again the analysis of the general case is hampered by the sheer complexity of the problem. Subject to a special choice of a Darboux sympletic atlas, the results indicate no obstructions to carrying out such duality maps.

In the sixth chapter, there is presented an indirect study of the CNM sigma- models via the use of duality with respect to CC models. This allows a direct inference of the constraints of the CNM model by using the duality of their correspondence to objects that occur in the CC approach.

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We include a chapter with our conclusions and include two appendices. The first appendix is used to state the conventions of the paper. The second contains explicit calculations of the actions that involve the CNM formulation to obtain component level results.

2. 6D, $\mathcal{N} = (1,0)$ CC sigma-models

We formulate six–dimensional nonlinear sigma–models using a formalism which keeps four dimensional $\mathcal{N} = 1$ supersymmetry[1] manifest.

The 6D, $\mathcal{N} = (1,0)$ hypermultiplet can be described in terms of two chiral multiplets [5, 6] (CC formulation) or one chiral multiplet and one complex linear multiplet [7] (CNM formulation). We start considering the CC formulation.

The action which describes the free dynamics of a 6D, $\mathcal{N} = (1,0)$ CC hypermultiplet [5 − 7] is

$$S_{CC} = \int d^6x\, d^4\theta \left[\, \overline{\Phi}_+ \Phi_+ + \overline{\Phi}_- \Phi_- \,\right] + \int d^6x\, d^2\theta \left[\, \Phi_+ \partial \Phi_- \,\right] + \int d^6x\, d^2\overline{\theta} \left[\, \overline{\Phi}_+ \overline{\partial}\, \overline{\Phi}_- \,\right] \quad , \tag{2.1}$$

where

$$\begin{aligned} z &\equiv \frac{1}{2}(x_4 + ix_5) \quad , \quad \partial \equiv \frac{\partial}{\partial z} = \partial_4 - i\partial_5 \quad ; \\ \overline{z} &\equiv \frac{1}{2}(x_4 - ix_5) \quad , \quad \overline{\partial} \equiv \frac{\partial}{\partial \overline{z}} = \partial_4 + i\partial_5 \quad . \end{aligned} \tag{2.2}$$

The action (2.1) is explicitly invariant under $\mathrm{Sl}(2,\mathbb{C})\times\mathrm{U}(1) \simeq \mathrm{SO}(1,3)\times\mathrm{SO}(2) \subset \mathrm{SO}(1,5)$, a proper subgroup of the 6D Lorentz group, and it has off–shell 4D, $\mathcal{N} = 1$ SUSY. The $\mathrm{U}(1) \simeq \mathrm{SO}(2)$ is the subgroup of rotations on the $(4,5)$-plane in 6D Minkowski space and acts as phase transformations on $\partial \to e^{i\phi}\partial$, $\overline{\partial} \to e^{-i\phi}\overline{\partial}$. The (anti)chiral superfields of the hypermultiplet are assumed to be neutral under the U(1) subgroup since the bosonic physical fields $A_\pm = \Phi_\pm|$ and $\overline{A}_\pm = \overline{\Phi}_\pm|$ must be neutral (i.e. scalars with respect to the 6D Lorentz group). From the invariance of the holomorphic[2] terms in (2.1), it follows that the grassmannian differentials transform as $d\theta_\alpha \to e^{-\frac{i}{2}\phi} d\theta_\alpha$, $d\overline{\theta}_{\dot\alpha} \to e^{\frac{i}{2}\phi} d\overline{\theta}_{\dot\alpha}$.

Once integrated out, the auxiliary fields in (2.1) lead to a resulting action which has linearly realized 6D Lorentz invariance and is on–shell 6D, $\mathcal{N} = (1,0)$ supersymmetric.

We now extend this analysis to 6D nonlinear sigma–models and find restrictions on the target space geometry induced by the request for the model to be 6D covariant and supersymmetric.

We start generalizing the action (2.1) to a system of n decoupled CC hypermultiplets describing a flat complex $2n$–dimensional target space. Defining $\Psi^a = (\Phi^I_+, \Phi^i_-)$ we write

$$S = \int d^6x \left[\int d^4\theta\, \overline{\Psi}^{\overline{a}} \delta_{\overline{a}b} \Psi^b + \frac{1}{2} \int d^2\theta\, \Psi^a\, \Omega_{ab}\, \partial\, \Psi^b + \frac{1}{2} \int d^2\overline{\theta}\, \overline{\Psi}^{\overline{a}}\, \overline{\Omega}_{\overline{a}\overline{b}}\, \overline{\partial}\, \overline{\Psi}^{\overline{b}} \right] \quad , \tag{2.3}$$

[1]We use the conventions of [20] and [7].

[2]We use ' holomorphic terms' instead of 'superpotential terms' since they lead to the appearance of derivatives of the propagating bosons.

where

$$\delta_{\overline{a}b} = \begin{pmatrix} \delta_{\overline{I}J} & 0 \\ 0 & \delta_{\overline{i}j} \end{pmatrix} \qquad\qquad \Omega_{ab} = \overline{\Omega}_{\overline{a}\overline{b}} = \begin{pmatrix} 0 & \delta_{Ij} \\ -\delta_{iJ} & 0 \end{pmatrix} . \tag{2.4}$$

To extend non–trivially the action (2.3) to a curved target space we make the following ansatz

$$\int d^6x \left[\int d^4\theta \; K\left(\Psi^a, \overline{\Psi}^{\overline{a}}\right) + \int d^2\theta \; Q_a\left(\Psi^b\right) \partial \, \Psi^a + \int d^2\overline{\theta} \; \overline{Q}_{\overline{a}}\left(\overline{\Psi}^{\overline{b}}\right) \overline{\partial} \, \overline{\Psi}^{\overline{a}} \right] \quad . \tag{2.5}$$

Here the functions Q_a ($\overline{Q}_{\overline{a}}$) are (anti)holomorphic in the (anti)chiral superfields Ψ^a ($\overline{\Psi}^{\overline{a}}$). The expression (2.5) is the most general ansatz for an action local in the physical fields which generalizes (2.3) and still has the off–shell symmetries of the flat case, i.e. 4D SUSY and the $\mathrm{Sl}(2,\mathbb{C}) \times \mathrm{U}(1)$ invariance.

A feature of note regarding (2.5) is the appearance of $Q_a(\Psi^b)$ in the extra–dimensions derivatives holomorphic term. This quantity has an interpretation as the connection of a U(1)-bundle. This U(1)-bundle is not necessarily related to the one that is part of $\mathrm{Sl}(2,\mathbb{C}) \times \mathrm{U}(1)$ invariance. In fact, the U(1)-bundle for which $Q_a(\Psi^b)$ is the connection is a bundle defined over the manifold. The fact that $Q_a(\Psi^b)$ appears as it does in (2.5) implies that it is ambiguous with respect the gauge transformation

$$Q_a(\Psi^b) \;\; \to \;\; Q_a(\Psi^b) \; + \; \frac{\partial}{\partial \Psi^a} \mathcal{T}(\Psi^b) \quad , \tag{2.6}$$

since the purely holomorphic terms are only changed by surface terms with regard to this redefinition. This invariance will be seen at the level of the action by the result that this U(1)-bundle connection will only appear in quantities via its exterior derivative.

It is important to note that the rigid U(1) invariance and as well the local manifold U(1)-bundle invariance, both fix the form of the the latter two terms in (2.5) and exclude the possibility to have terms like $\int d^2\theta \widetilde{Q}_a \overline{\partial} \, \Psi^a + \int d^2\overline{\theta} \widetilde{\overline{Q}}_{\overline{a}} \, \partial \, \overline{\Psi}^{\overline{a}}$. In analogy with the flat space [7], such contributions would be the only possible terms admitted if we were to impose opposite U(1) phase transformations on the grassmanian coordinates of the 4D, $\mathcal{N} = 1$ superspace and would give $\mathcal{N} = (0,1)$ CC sigma–models. In the rest of the paper we concentrate only on the $(1,0)$ case. As noted in [7], the $(0,1)$ case can be recovered by simply doing the change $\partial \leftrightarrow -\overline{\partial}$ wherever ∂ and $\overline{\partial}$ appear.

Reduced in components the action (2.5) reads

$$\begin{aligned} \int d^6x \Bigg\{ & K_{a\overline{b}} \left[-\frac{1}{2} \, \partial^{\alpha\dot{\alpha}} \overline{A}^{\overline{b}} \partial_{\alpha\dot{\alpha}} A^a + \overline{F}^{\overline{b}} F^a - \frac{i}{2} \left(\overline{\psi}^{\overline{b}}_{\dot{\alpha}} \partial^{\alpha\dot{\alpha}} \psi^a_\alpha + \psi^a_\alpha \partial^{\alpha\dot{\alpha}} \overline{\psi}^{\overline{b}}_{\dot{\alpha}} \right) \right] \\ & + \frac{1}{2} \, K_{ab\overline{c}} \left[\overline{F}^{\overline{c}} \psi^{a\alpha} \psi^b_\alpha + i (\partial^{\alpha\dot{\alpha}} A^b) \psi^a_\alpha \overline{\psi}^{\overline{c}}_{\dot{\alpha}} \right] \\ & + \frac{1}{2} \, K_{c\overline{a}\overline{b}} \left[F^c \overline{\psi}^{\overline{a}\dot{\alpha}} \overline{\psi}^{\overline{b}}_{\dot{\alpha}} - i (\partial^{\alpha\dot{\alpha}} \overline{A}^{\overline{b}}) \psi^c_\alpha \overline{\psi}^{\overline{a}}_{\dot{\alpha}} \right] \\ & + Q_{a(b)} \psi^{b\alpha} \partial \, \psi^a_\alpha + \Big(Q_{b(a)} - Q_{a(b)} \Big) F^a \partial A^b \; + \frac{1}{2} \, Q_{a(bc)} \left(\partial A^a \right) \psi^{b\alpha} \psi^c_\alpha \\ & + \overline{Q}_{\overline{a}(\overline{b})} \overline{\psi}^{\overline{b}\dot{\alpha}} \overline{\partial} \, \overline{\psi}^{\overline{a}}_{\dot{\alpha}} + \Big(\overline{Q}_{\overline{b}(\overline{a})} - \overline{Q}_{\overline{a}(\overline{b})} \Big) \overline{F}^{\overline{a}} \overline{\partial} \, \overline{A}^{\overline{b}} + \frac{1}{2} \, \overline{Q}_{\overline{a}\,(\overline{b}\overline{c})} (\overline{\partial} \, \overline{A}^{\overline{a}}) \, \overline{\psi}^{\overline{b}\dot{\alpha}} \overline{\psi}^{\overline{c}}_{\dot{\alpha}} \end{aligned}$$

$$+\frac{1}{4}\,K_{ab\overline{a}\overline{b}}\,\psi^{a\alpha}\psi^b_\alpha\overline{\psi}^{\overline{a}\dot{\alpha}}\overline{\psi}^{\overline{b}}_{\dot{\alpha}}\Bigg\}\quad, \tag{2.7}$$

where we have defined the tensors

$$K_{a_1\cdots a_p\overline{b}_1\cdots\overline{b}_q}\equiv\frac{\partial^{p+q}K(A,\overline{A})}{\partial A^{a_1}\cdots\partial A^{a_p}\partial\overline{A}^{\overline{b}_1}\cdots\partial\overline{A}^{\overline{b}_q}}\quad, \tag{2.8}$$

$$Q_{a(b_1\cdots b_r)}\equiv\frac{\partial^r Q_a(A)}{\partial A^{b_1}\cdots\partial A^{b_r}}\quad,\quad\overline{Q}_{\overline{a}(\overline{b}_1\cdots\overline{b}_r)}\equiv\frac{\partial^r\overline{Q}_{\overline{a}}(\overline{A})}{\partial\overline{A}^{\overline{b}_1}\cdots\partial\overline{A}^{\overline{b}_r}}\quad. \tag{2.9}$$

The equations of motion for the auxiliary F–fields are algebraic as in the free case

$$F^a=-K^{a\overline{b}}\left[\frac{1}{2}\,K_{cd\overline{b}}\,\psi^{c\alpha}\psi^d_\alpha+\Big(\overline{Q}_{\overline{c}(\overline{b})}-\overline{Q}_{\overline{b}(\overline{c})}\Big)\overline{\partial}\,\overline{A}^{\overline{c}}\right]\quad, \tag{2.10}$$

$$\overline{F}^{\overline{a}}=-K^{b\overline{a}}\left[\frac{1}{2}\,K_{b\overline{c}\overline{d}}\,\overline{\psi}^{\overline{c}\dot{\alpha}}\overline{\psi}^{\overline{d}}_{\dot{\alpha}}+\Big(Q_{c(b)}-Q_{b(c)}\Big)\partial A^c\right]\quad, \tag{2.11}$$

where $K^{a\overline{b}}$ is the inverse of the Kähler metric $K_{a\overline{b}}$, $K_{a\overline{c}}K^{b\overline{c}}=\delta^b_a$ and $K_{c\overline{a}}K^{c\overline{b}}=\delta^{\overline{b}}_{\overline{a}}$. Inserting the previous relations in (2.7) we find the action for the physical component fields. We divide it into three pieces with zero, two and four fermionic fields, respectively

$$S_{0f}=\int d^6x\left[-\frac{1}{2}\,K_{a\overline{a}}\,\partial^{\alpha\dot{\alpha}}\overline{A}^{\overline{a}}\partial_{\alpha\dot{\alpha}}A^a-K^{a\overline{a}}\Big(Q_{b(a)}-Q_{a(b)}\Big)\Big(\overline{Q}_{\overline{b}(\overline{a})}-\overline{Q}_{\overline{a}(\overline{b})}\Big)\overline{\partial}\,\overline{A}^{\overline{b}}\partial A^b\right]\quad, \tag{2.12}$$

$$\begin{aligned}S_{2f}=-\frac{1}{2}\int d^6x\Bigg[&K_{a\overline{a}}\,\overline{\psi}^{\overline{a}}_{\dot{\alpha}}i\partial^{\alpha\dot{\alpha}}\psi^a_\alpha+K_{ab\overline{b}}\,\Big(i\partial^{\alpha\dot{\alpha}}A^a\Big)\overline{\psi}^{\overline{b}}_{\dot{\alpha}}\psi^b_\alpha+\Big(Q_{b(a)}-Q_{a(b)}\Big)\psi^{b\alpha}\partial\,\psi^a_\alpha\\&+K^{a\overline{a}}K_{bc\overline{a}}\Big(Q_{d(a)}-Q_{a(d)}\Big)\Big(\partial A^d\Big)\psi^{b\alpha}\psi^c_\alpha\\&+\Big(Q_{b(ac)}-Q_{a(bc)}\Big)\Big(\partial A^a\Big)\psi^{b\alpha}\psi^c_\alpha+\ \{\text{h. c.}\}\Bigg]\quad,\end{aligned} \tag{2.13}$$

$$S_{4f}=\frac{1}{4}\int d^6x\left[\Big(K_{ab\overline{a}\overline{b}}\ -\ K^{c\overline{c}}K_{ab\overline{c}}K_{c\overline{a}\overline{b}}\Big)\psi^{a\alpha}\psi^b_\alpha\overline{\psi}^{\overline{a}\dot{\alpha}}\overline{\psi}^{\overline{b}}_{\dot{\alpha}}\right]\quad. \tag{2.14}$$

In these actions the structures of the Kähler geometry as required by manifest 4D, $\mathcal{N}=1$ SUSY appear: In (2.13), (2.14), besides the metric, we recognize the connections and the curvature tensor of the Kähler manifold

$$\begin{aligned}\Gamma^a_{\ bc}=K^{a\overline{d}}K_{bc\overline{d}}\quad,\quad\Gamma^{\overline{a}}_{\ \overline{b}\overline{c}}=K^{d\overline{a}}K_{d\overline{b}\overline{c}}\quad,\\ \mathcal{R}_{a\overline{b}c\overline{d}}=K_{ac\overline{b}\overline{d}}\ -\ K^{r\overline{s}}K_{ac\overline{s}}K_{r\overline{b}\overline{d}}\quad.\end{aligned} \tag{2.15}$$

Extra constraints on the geometrical structures come from requiring that after integration on the auxiliary F–fields, the resulting action is 6D Lorentz invariant. In particular the actions (2.12), (2.13), (2.14) must be separately Lorentz invariant.

Bosonic action We start imposing 6D Lorentz symmetry for the pure bosonic action (2.12). In order to have manifest, linearly realized 6D Lorentz invariance we should be able to write it as

$$-\int d^6x \left[K_{a\overline{a}}\,\partial^{\mu}\overline{A}^{\overline{a}}\partial_{\mu}A^a \right] = -\frac{1}{2}\int d^6x \left[K_{a\overline{a}}\Big(\partial^{\alpha\dot{\alpha}}\overline{A}^{\overline{a}}\partial_{\alpha\dot{\alpha}}A^a + \overline{\partial}\,\overline{A}^{\overline{a}}\partial A^a + \partial\overline{A}^{\overline{a}}\overline{\partial}A^a\Big)\right] . \tag{2.16}$$

To compare the action (2.12) with (2.16) we re-write (2.12) as

$$\begin{aligned} S_{0f} \;=\; -\frac{1}{2}\int d^6x \bigg[& K_{a\overline{a}}\,\partial^{\alpha\dot{\alpha}}\overline{A}^{\overline{a}}\partial_{\alpha\dot{\alpha}}A^a + \widetilde{K}_{a\overline{a}}\left(\overline{\partial}\,\overline{A}^{\overline{a}}\partial A^a + \partial\overline{A}^{\overline{a}}\overline{\partial}A^a\right) \\ & + \widetilde{K}_{a\overline{a}}\left(\overline{\partial}\,\overline{A}^{\overline{a}}\partial A^a - \partial\overline{A}^{\overline{a}}\overline{\partial}A^a\right)\bigg] \quad , \end{aligned} \tag{2.17}$$

where we have defined

$$\widetilde{K}_{a\overline{a}} \;\equiv\; \Big(Q_{b(a)} - Q_{a(b)}\Big)\,K^{b\overline{b}}\left(\overline{Q}_{\overline{b}(\overline{a})} - \overline{Q}_{\overline{a}(\overline{b})}\right) \;\equiv\; -\Omega_{ab}\,K^{b\overline{b}}\,\overline{\Omega}_{\overline{b}\overline{a}} \quad , \tag{2.18}$$

and

$$\Omega_{ab} \;\equiv\; \Big(Q_{b(a)} - Q_{a(b)}\Big) \quad , \quad \overline{\Omega}_{\overline{a}\overline{b}} \;\equiv\; \Big(\overline{Q}_{\overline{b}(\overline{a})} - \overline{Q}_{\overline{a}(\overline{b})}\Big) \quad . \tag{2.19}$$

Matching (2.17) with (2.16) requires

$$K_{a\overline{a}} \;=\; \widetilde{K}_{a\overline{a}} \;=\; -\Omega_{ab}\,K^{b\overline{b}}\,\overline{\Omega}_{\overline{b}\overline{a}} \quad . \tag{2.20}$$

The second line of (2.17) then becomes

$$\begin{aligned} & -\frac{1}{2}\int d^6x \Big(K_{a\overline{a}}\,\overline{\partial}\,\overline{A}^{\overline{a}}\partial A^a + K_{ab}\,\overline{\partial}A^b\partial A^a - K_{ab}\,\partial A^b\overline{\partial}A^a - K_{a\overline{a}}\,\partial\overline{A}^{\overline{a}}\,\overline{\partial}A^a\Big) \\ & = -\frac{1}{2}\int d^6x \Big[\overline{\partial}\left(K_a\partial A^a\right) - \partial\left(K_a\overline{\partial}A^a\right)\Big] \quad , \end{aligned} \tag{2.21}$$

and it is explicitly a total derivative in six dimensions.

An interesting observation regarding the total derivative term is that if we were to work in 5D [22] the second line of (2.17) would be identically zero, even without imposing $K_{a\overline{a}} = \widetilde{K}_{a\overline{a}}$ since $\partial = \overline{\partial}$ in five dimensions.

To summarize, in order to have 6D Lorentz invariance of (2.12) we need only require the constraint (2.20) for the Kähler metric. Note that if we interpret Q_a as a holomorphic 1-form connection for a U(1) bundle then clearly the quantity Ω_{ab} is its exterior derivative (i.e. its field strength - U(1) curvature).

We now study the consequences of the constraint (2.20). Taking the determinant of both sides of (2.20) we have an expression that relates the determinant of the Kähler metric to the exterior derivative of the holomorphic one-form

$$\begin{aligned} \det K = \det\Omega\,\det K^{-1}\det\overline{\Omega} \;\Longrightarrow\; & \Big[\det K\Big]^2 \;=\; \det\Omega\,\det\overline{\Omega} = |\det\Omega\,|^2 \\ \Longrightarrow\quad \mathrm{Tr}[\ln(K)\,] \;=\; & \tfrac{1}{2}\Big[\mathrm{Tr}[\ln(\Omega)\,] \;+\; \mathrm{Tr}[\ln(\overline{\Omega})\,]\Big] \quad . \end{aligned} \tag{2.22}$$

These equations can be re-written in terms of the Kähler potential as a nonlinear d-th order differential equation

$$[\det(\partial_a \partial_{\overline{b}} K)] \; = \; |\det \Omega| \tag{2.23}$$

$$\tfrac{1}{d!}\, \epsilon^{a_1\, a_2 \ldots\, a_d}\, \epsilon^{\overline{b}_1\, \overline{b}_2\, \ldots\, \overline{b}_d}\, (\partial_{a_i} \partial_{\overline{b}_i} K) \cdots (\partial_{a_i} \partial_{\overline{b}_i} K) \; = \; |\det \Omega| \quad , \tag{2.24}$$

where d is the number of the chiral doublets present in the action. After the introduction of a new variable $\mathcal{K}(\Phi, \overline{\Phi})$ via the equation $K = \Phi^a \overline{\Phi}_{\bar{a}} + \mathcal{K}$ this leads to a nonlinear differential equation for $\mathcal{K}$,

$$\det(\delta_{a\,\bar{b}} \; + \; \partial_a \partial_{\overline{b}} \mathcal{K}) \; = \; |\det \Omega| \quad . \tag{2.25}$$

Since our manifold is Kähler we can express the Ricci tensor in terms of the determinant of the metric as $R_{a\bar{b}} = \partial_a \partial_{\overline{b}}[\ln(\det K)]$ and from (2.22) it follows

$$R_{a\bar{b}} \; = \; \frac{1}{2}\, \partial_a \partial_{\overline{b}}\, [\ln(\det \Omega) + \ln(\det \overline{\Omega})] \; = \; 0 \quad . \tag{2.26}$$

Our manifold is then Ricci flat.

We note that, through a holomorphic change of coordinates, the $\det \Omega$ can be always chosen to be unimodular. Then the previous description (2.22)–(2.25) is equivalent to Monge-Ampère equation $\det(\partial_a \partial_{\overline{b}} K) = 1$ which characterizes the Ricci flatness of our target space.

It is known that relations (2.19), (2.20) imply moreover the stronger constraint on the target space geometry to be hyper-Kähler [23]. In fact, we introduce

$$\Omega^{ac}\, \Omega_{cb} \; = \; \delta^a_b \quad , \quad \overline{\Omega}^{\overline{ac}}\, \overline{\Omega}_{\overline{cb}} \; = \; \delta^{\overline{a}}_{\overline{b}} \tag{2.27}$$

and define

$$\Omega^{\overline{a}}{}_b \; \equiv \; K^{c\overline{a}}\, \Omega_{cb} \; = \; -\, \overline{\Omega}^{\overline{ac}} K_{b\overline{c}} \quad , \tag{2.28}$$

$$\Omega^{a}{}_{\overline{b}} \; \equiv \; K^{a\overline{c}}\, \overline{\Omega}_{\overline{cb}} \; = \; -\, \Omega^{ac} K_{c\overline{b}} \quad , \tag{2.29}$$

satisfying

$$\Omega^a{}_{\overline{c}}\, \Omega^{\overline{c}}{}_b \; = \; -\, \delta^a_b \quad , \quad \Omega^{\overline{a}}{}_c\, \Omega^c{}_{\overline{b}} \; = \; -\, \delta^{\overline{a}}_{\overline{b}} \quad . \tag{2.30}$$

It then follows

$$\partial_a\, \Omega_{bc} \; = \; K_{ab\overline{b}}\, \Omega^{\overline{b}}{}_c \; - \; K_{ac\overline{b}}\, \Omega^{\overline{b}}{}_b \quad \Longrightarrow \quad \nabla_a\, \Omega_{bc} \; = \; 0 \quad , \tag{2.31}$$

$$\partial_{\overline{a}}\, \Omega_{\overline{bc}} \; = \; K_{b\overline{ab}}\, \Omega^b{}_{\overline{c}} \; - \; K_{b\overline{ac}}\, \Omega^b{}_{\overline{b}} \quad \Longrightarrow \quad \nabla_{\overline{a}}\, \overline{\Omega}_{\overline{bc}} \; = \; 0 \quad , \tag{2.32}$$

and Ω_{ab}, $\overline{\Omega}_{\overline{ab}}$, $\Omega^a{}_{\overline{b}}$ and $\Omega^{\overline{a}}{}_b$ are covariantly constant. A triplet of covariantly constant complex structures can be then introduced as in [23]–[26]

$$J^1 = \begin{pmatrix} 0 & \Omega^a{}_{\overline{b}} \\ \Omega^{\overline{a}}{}_b & 0 \end{pmatrix} \quad , \quad J^2 = \begin{pmatrix} 0 & i\Omega^a{}_{\overline{b}} \\ -i\Omega^{\overline{a}}{}_b & 0 \end{pmatrix} \quad , \quad J^3 = \begin{pmatrix} i\delta^a{}_b & 0 \\ 0 & -i\delta^{\overline{a}}{}_{\overline{b}} \end{pmatrix} \quad . \tag{2.33}$$

which define the quaternionic structure of an hyper-Kähler manifold

$$J^\mu\, J^\nu \; = \; -\, \delta^{\mu\nu} \; + \; \epsilon^{\mu\nu\rho}\, J^\rho \quad , \tag{2.34}$$

Therefore, the request for the on–shell bosonic action to be 6D Lorentz invariant implies the target space to be hyper-Kähler.

Fermionic actions We now investigate the Lorentz invariance of the fermionic actions (2.13, 2.14). As we are going to prove, the hyper-Kähler condition for the target manifold is sufficient to automatically provide 6D Lorentz invariance also for the fermionic actions, once properly defined the 6D, $(1,0)$ spinors[3] as obtained from the 4D spinor components of the (anti)chiral superfields ($\overline{\Psi}^{\overline{a}}$) Ψ^a. The correct choice of 6D spinors is the one suggested by the dimensional reduction of [2] and used also in the recent five dimensional analogue of our investigation [22]

$$\Psi^{a\tilde{\alpha}} = \begin{pmatrix} \psi^{a\alpha} \\ \Omega^{a}{}_{\overline{b}}\overline{\psi}^{\overline{b}\dot{\alpha}} \end{pmatrix} \quad , \quad \overline{\Psi}^{\overline{a}\tilde{\alpha}} = \begin{pmatrix} -\Omega^{\overline{a}}{}_{b}\psi^{b\alpha} \\ \overline{\psi}^{\overline{a}\dot{\alpha}} \end{pmatrix} = -\Omega^{\overline{a}}{}_{b}\Psi^{b\tilde{\alpha}} \quad . \tag{2.35}$$

Note that this is also the choice that gives a symplectic Majorana–Weyl structure to the 6D spinor. In fact, $(\Psi^{a\tilde{\alpha}})^* = \overline{\Psi}^{\overline{a}\dot{\tilde{\alpha}}} = \mathcal{C}^{\dot{\tilde{\alpha}}}{}_{\tilde{\beta}}\,\Omega^{\overline{a}}{}_{b}\Psi^{b\tilde{\beta}}$ where $\mathcal{C}^{\tilde{\alpha}}{}_{\dot{\tilde{\beta}}}\mathcal{C}^{\dot{\tilde{\beta}}}{}_{\tilde{\gamma}} = -\delta^{\tilde{\alpha}}{}_{\tilde{\gamma}}$.
Now, using the following relations due to the hyper-Kähler structure

$$\mathcal{R}_{a\overline{a}b\overline{b}}\,\Omega^{b}{}_{\overline{c}} \;=\; \partial_a\Big(\Gamma^{\overline{d}}_{\overline{a}\overline{b}}\,\overline{\Omega}_{\overline{d}\overline{c}}\Big) \;=\; \partial_a\Big(\Gamma^{\overline{d}}_{\overline{a}\overline{c}}\,\overline{\Omega}_{\overline{d}\overline{b}}\Big) \;=\; \mathcal{R}_{a\overline{a}b\overline{c}}\,\Omega^{b}{}_{\overline{b}} \quad , \tag{2.36}$$

$$\mathcal{R}_{a\overline{a}b\overline{b}}\,\Omega^{\overline{b}}{}_{c} \;=\; \partial_{\overline{a}}\Big(\Gamma^{d}{}_{ab}\,\Omega_{dc}\Big) \;=\; \partial_{\overline{a}}\Big(\Gamma^{d}{}_{ac}\,\Omega_{db}\Big) \;=\; \mathcal{R}_{a\overline{a}c\overline{b}}\,\Omega^{\overline{b}}{}_{b} \quad , \tag{2.37}$$

we find that the two and four fermions actions (2.13, 2.14) can be re-written as

$$\begin{aligned} S_{2f} \;=&\; -\frac{1}{2}\int d^6x\, K_{a\overline{a}}\Big[\,\overline{\psi}^{\overline{a}}_{\dot{\alpha}}\, i\partial^{\alpha\dot{\alpha}}\psi^{a}_{\alpha} \;+\; \Gamma^{a}_{bc}\Big(i\partial^{\alpha\dot{\alpha}}A^{c}\Big)\,\overline{\psi}^{\overline{b}}_{\dot{\alpha}}\psi^{b}_{\alpha} \\ &\qquad +\; \Omega^{\overline{a}}{}_{b}\,\psi^{b\alpha}\partial\psi^{a}_{\alpha} \;+\; \Gamma^{a}_{cd}\Big(\partial A^{d}\Big)\,\Omega^{\overline{a}}{}_{b}\,\psi^{b\alpha}\psi^{c}_{\alpha} \;+\; \{\,\mathrm{h.\,c.}\,\}\Big] \\ =&\; \frac{1}{4}\int d^6x\, K_{a\overline{a}}\Big[\,\overline{\Psi}^{\overline{a}\tilde{\alpha}} i\partial_{\tilde{\alpha}\tilde{\beta}}\Psi^{a\tilde{\beta}} \;+\; \overline{\Psi}^{\overline{a}\tilde{\alpha}}\Gamma^{a}_{bc}\Big(i\partial_{\tilde{\alpha}\tilde{\beta}}A^{b}\Big)\,\Psi^{c\tilde{\beta}} \\ &\qquad +\; \Psi^{a\tilde{\beta}} i\partial_{\tilde{\alpha}\tilde{\beta}}\overline{\Psi}^{\overline{a}\tilde{\alpha}} \;+\; \Psi^{a\tilde{\beta}}\Gamma^{\overline{a}}_{\overline{b}\overline{c}}\Big(i\partial_{\tilde{\alpha}\tilde{\beta}}\overline{A}^{\overline{b}}\Big)\,\overline{\Psi}^{\overline{c}\tilde{\alpha}}\Big] \quad , \end{aligned} \tag{2.38}$$

$$S_{4f} \;=\; -\frac{1}{24}\int d^6x\; \mathcal{R}_{a\overline{a}b\overline{b}}\,\epsilon_{\tilde{\alpha}\tilde{\beta}\tilde{\gamma}\tilde{\delta}}\;\Psi^{a\tilde{\alpha}}\,\Psi^{b\tilde{\beta}}\,\overline{\Psi}^{\overline{a}\tilde{\gamma}}\,\overline{\Psi}^{\overline{b}\tilde{\delta}} \quad . \tag{2.39}$$

and Lorentz invariance become manifest. We have then found that 6D Lorentz invariance requires the target space to be hyper-Kähler. Under this condition, the sum of actions $(S_{0f} + S_{2f} + S_{4f})$ is also on–shell $\mathcal{N} = (1,0)$ supersymmetric [2].

Our sigma–model, being written in 4D $\mathcal{N} = 1$ superspace, has manifest 4D supersymmetry. When the hyper-Kähler conditions are satisfied, the action is also on–shell invariant under the following transformations

$$\delta_{\eta_2}\Psi^a = \overline{D}^2\Big[\,\Omega^{ab}K_b\,(\theta^\alpha\eta_{2\alpha} + \overline{\theta}^{\dot{\alpha}}\overline{\eta}_{2\dot{\alpha}})\Big] \quad , \quad \delta_{\eta_2}\overline{\Psi}^{\overline{a}} = D^2\Big[\,\overline{\Omega}^{\overline{a}\overline{b}}K_{\overline{b}}\,(\theta^\alpha\eta_{2\alpha} + \overline{\theta}^{\dot{\alpha}}\overline{\eta}_{2\dot{\alpha}})\Big] \,. \tag{2.40}$$

The 6D, $\mathcal{N} = (1,0)$ algebra, once written in a 4D formalism, is equivalent to a 4D, $\mathcal{N} = 2$ SUSY algebra with a complex central charge [7]. The transformations (2.40) give exactly

[3] for our $(1,0)$ spinor conventions see [7].

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the second supersymmetry of the 4D, $\mathcal{N}=2$ algebra. In fact, it can be seen that the commutator of two transformations $[\delta_{\eta_2}, \delta_{\zeta_2}]\Psi^a$ closes off–shell, and the commutator of a transformation (2.40) with a 4D $\mathcal{N}=1$ transformation closes on–shell as $[\delta_{\eta_2}, \delta_{\zeta_1}]\Psi^a = \partial\,\Psi^a(\zeta_1^\alpha\eta_{2\alpha}+\overline{\zeta}_1^{\dot{\alpha}}\overline{\eta}_{2\dot{\alpha}})$ and $[\delta_{\eta_2}, \delta_{\zeta_1}]\overline{\Psi}^{\overline{a}} = \overline{\partial}\,\overline{\Psi}^{\overline{a}}(\zeta_1^\alpha\eta_{2\alpha}+\overline{\zeta}_1^{\dot{\alpha}}\overline{\eta}_{2\dot{\alpha}})$ on the extra dimensions. These properties are the natural extension to six dimensions of what happens for five–dimensional CC sigma–models [22].

3. 6D sigma-models from projective superspace

Up to now we have studied 6D supersymmetric sigma–models using a partially on–shell formalism which keeps 4D, $\mathcal{N}=1$ SUSY manifest, being the target space coordinates described by 4D (anti)chiral superfield. This description is convenient due to the simplicity of the 4D, $\mathcal{N}=1$ superspace structures but it has the disadvantage to realize only on–shell invariance under the whole 6D superpoincaré group.

If we are interested in off–shell 6D superpoincaré invariant formulations, the most powerful description is harmonic superspace [27]–[29] with eight supercharges which realize 6D, $\mathcal{N}=1$ SUSY and SU(2) automorphism group. However, as we have emphasized previously, such constructions and approaches, at the quantum level, are necessarily bedeviled with harmonic divergences that make higher loop calculations ambiguous. Indeed, there presently does not exist a proof that such ambiguities can be removed to all orders of perturbation theory.

An alternative formulation which guarantees manifest off–shell supersymmetry for theories with eight supercharges can be obtained by using the projective superspace technique [9]–[13]. The two off–shell formulations are strictly related [12] and the main difference is that the projective superspace approach has only a U(1) subgroup linearly realized, out of the SU(2) automorphism. The interesting property of projective superspace is that it naturally provides a reduction to 4D, $\mathcal{N}=1$ superspace which the harmonic approach does not admit.

Since in this paper we are interested in studying properties of 6D supersymmetric sigma–models with target space geometry parametrized by 4D, $\mathcal{N}=1$ superfields, the projective superspace approach seems to be the most natural one. A similar analysis has been recently performed for the 5D case in a series of papers [14].

We start reviewing the definitions and properties of projective superspace in 6D [15, 7]. We focus on the reduction to 4D, $\mathcal{N}=1$ superspace following the lines of our recent paper [7] (For conventions we refer the reader to this reference).

The algebra of the $\mathcal{N}=(1,0)$ supercovariant derivatives is

$$\{D^{a\tilde{\alpha}}, D^{b\tilde{\beta}}\} = \epsilon^{ab} i\partial^{\tilde{\alpha}\tilde{\beta}} \quad , \tag{3.1}$$

where ϵ^{ab} is the invariant tensor of the SU(2) automorphism group of the $\mathcal{N}=(1,0)$ algebra and the derivatives $D^{a\tilde{\alpha}}$ are $(1,0)$ Weyl spinors satisfying a SU(2)–Majorana condition [1]. Now we extend the 6D superspace parametrized by $Z=(x^\mu, \theta_{a\tilde{\alpha}})$ with a projective complex variable $\zeta \in \mathbb{C}^*$. In analogy with the 4D case we define the projective supercovariant derivatives as

$$\nabla^{\tilde{\alpha}}(\zeta)=u_a\nabla^{a\tilde{\alpha}} \quad , \quad \Delta^{\tilde{\alpha}}(\zeta)=v_a\nabla^{a\tilde{\alpha}} \quad ; \quad u_a=(1,\zeta) \quad , \quad v_a=\left(-\frac{1}{\zeta},1\right) \quad , \tag{3.2}$$

satisfying

$$\{\nabla^{\tilde{\alpha}},\nabla^{\tilde{\beta}}\}=0 \quad , \quad \{\Delta^{\tilde{\alpha}},\Delta^{\tilde{\beta}}\}=0 \quad , \quad \{\nabla^{\tilde{\alpha}},\Delta^{\tilde{\beta}}\}=-2i\partial^{\tilde{\alpha}\tilde{\beta}} \quad . \tag{3.3}$$

We define superfields living in projective superspace as superfields holomorphic in ζ

$$\Xi(Z,\zeta)=\sum_{n=-\infty}^{+\infty}\Xi_n(Z)\zeta^n \quad , \tag{3.4}$$

and satisfying

$$\nabla^{\tilde{\alpha}}\,\Xi(Z,\zeta)=0 \quad . \tag{3.5}$$

Following Ref. [7] we want to make the structures of 4D superfields manifest. In terms of 4D spinorial coordinates the 6D superspace is parametrized by $Z=(x^\mu,\theta^{a\alpha},\overline{\theta}^{\dot{\alpha}}_a)$ and the algebra (3.1) is rewritten as

$$\{D_{a\alpha},D_{b\beta}\}=\epsilon_{ab}C_{\alpha\beta}\overline{\partial} \quad , \quad \{\overline{D}^a_{\dot{\alpha}},\overline{D}^b_{\dot{\beta}}\}=\epsilon^{ab}C_{\dot{\alpha}\dot{\beta}}\partial \quad , \quad \{D_{a\alpha},\overline{D}^b_{\dot{\beta}}\}=\delta^b_a i\partial_{\alpha\dot{\beta}} \quad . \tag{3.6}$$

It is interesting to note that this is equivalent to the algebra of 4D, $\mathcal{N}=2$ SUSY with a complex central charge [20]. In 4D notations, the projective supercovariant derivatives are

$$\nabla^{\tilde{\alpha}}=\begin{pmatrix}\nabla^\alpha\\ \overline{\nabla}^{\dot{\alpha}}\end{pmatrix}=\begin{pmatrix}\zeta D^\alpha_1-D^\alpha_2\\ \overline{D}^{1\dot{\alpha}}+\zeta\overline{D}^{2\dot{\alpha}}\end{pmatrix} \quad , \quad \Delta^{\tilde{\alpha}}=\begin{pmatrix}\Delta^\alpha\\ \overline{\Delta}^{\dot{\alpha}}\end{pmatrix}=\begin{pmatrix}D^\alpha_1+\frac{1}{\zeta}D^\alpha_2\\ \overline{D}^{2\dot{\alpha}}-\frac{1}{\zeta}\overline{D}^{1\dot{\alpha}}\end{pmatrix} \quad . \tag{3.7}$$

Then, from the definition (3.5), projective superfields satisfy

$$\nabla_\alpha(\zeta)\Xi=0=\overline{\nabla}_{\dot{\alpha}}(\zeta)\Xi \quad \Longleftrightarrow \quad D_{2\alpha}\Xi=\zeta D_{1\alpha}\Xi \quad , \quad \overline{D}^1_{\dot{\alpha}}\Xi=-\zeta\overline{D}^2_{\dot{\alpha}}\Xi \quad , \tag{3.8}$$

and the component superfields (3.4) are constrained by

$$D_{2\alpha}\Xi_{n+1}=D_{1\alpha}\Xi_n \quad , \quad \overline{D}^2_{\dot{\alpha}}\Xi_n=-\overline{D}^1_{\dot{\alpha}}\Xi_{n+1} \quad . \tag{3.9}$$

The above constraints fix the dependence of the Ξ_n on half of the Grassmannian coordinates of the superspace. The superfields Ξ_n can then be considered as superfields living on a $\mathcal{N}=1$ superspace with $\theta^\alpha=\theta^{1\alpha}$, $\overline{\theta}^{\dot{\alpha}}=\overline{\theta}^{\dot{\alpha}}_1$ [9]–[13], [7] and we have a natural reduction of 6D, $\mathcal{N}=(1,0)$ multiplets to 4D, $\mathcal{N}=1$ superfields.
In projective superspace the natural conjugation operation combines complex conjugation with the antipodal map on the Riemann sphere ($\zeta\to-1/\zeta$) and acts on projective superfields as

$$\breve{\Xi}=\sum_{n=-\infty}^{+\infty}\breve{\Xi}_n\,\zeta^n=\sum_{n=-\infty}^{+\infty}(-1)^n\overline{\Xi}_{-n}\,\zeta^n \quad . \tag{3.10}$$

Defining $\Delta^4 = \frac{1}{24}\epsilon_{\tilde\alpha\tilde\beta\tilde\gamma\tilde\delta}\Delta^{\tilde\alpha}\Delta^{\tilde\beta}\Delta^{\tilde\gamma}\Delta^{\tilde\delta}$, manifestly 6D $\mathcal{N}=(1,0)$ SUSY invariant actions have the general form[4] [15, 7]

$$-\int d^6x \left\{ \oint_C \frac{\zeta d\zeta}{32\pi i}\, \Delta^4\, \mathcal{L}(\Xi, \breve{\Xi}, \zeta)\Big| \right\} = \int d^6x d^4\theta \left\{ \oint_C \frac{d\zeta}{2\pi i\zeta}\, \mathcal{L}(\Xi, \breve{\Xi}, \zeta) \right\} \quad , \tag{3.11}$$

where $\mathcal{L}(\Xi, \breve{\Xi}, \zeta)$ is real under the $\smile$-conjugation of (3.10) and C is a contour around the origin of the complex ζ–plane.

The general classification of multiplets in projective superspace is based on the analyticity properties of the projective superfields in the ζ–plane [9]–[12] and it is essentially not affected by the dimensions of the space–time. What different dimensions affect is the original SUSY algebra with eight supercharges which are used to define the projective superspace. Note that the 6D case is interesting in this regard, six being the largest dimension in which hypermultiplets with only $(0, \frac{1}{2})$ degrees of freedom can be defined. Therefore, it can be considered as the parent (up to issues involving 'twists' and such dualities) of all lower dimensional theories with only $(0, \frac{1}{2})$ multiplets constructed by dimensional reduction.

Now, we consider a particular class of examples built using the 6D polar multiplet [7] defined by (ant)artic superfields focusing on the reduction from projective superfields to 4D, $\mathcal{N}=1$ superfields degrees of freedom. It is an interesting feature of the 4D and 5D projective superfields to provide coordinates for natural extensions of rigid $\mathcal{N}=1$ Kähler nonlinear sigma–models to the $\mathcal{N}=2$ cases [13, 14]. Adapting these extensions to the 6D projective superspace it is straightforward to find the same geometrical structures.

We start by considering a 4D $\mathcal{N}=1$ rigid supersymmetric sigma–model [30]

$$\int d^4xd^4\theta\ K(\Phi^I, \overline{\Phi}^{\overline{I}}) \quad , \tag{3.12}$$

with K the Kähler potential of the target space Kähler manifold $\mathcal{M}$ parametrized by the scalar components of Φ^I ($\overline{\Phi}^{\overline{I}}$). In analogy to the 4D case we define a 6D $\mathcal{N}=(1,0)$ sigma–model on $\mathcal{M}$

$$\int d^6xd^4\theta \left\{ \oint_C \frac{d\zeta}{2\pi i\zeta} K(\Upsilon^I(\zeta), \breve{\Upsilon}^{\overline{I}}(\zeta)) \right\} \quad . \tag{3.13}$$

where K is a function of the 6D (ant)artic projective superfields ($\breve{\Upsilon}^{\overline{I}}$) Υ^I defined by the following power series

$$\Upsilon^I = \sum_{n=0}^{+\infty} \Upsilon^I_n \zeta^n \ , \ \breve{\Upsilon}^{\overline{I}} = \sum_{n=0}^{+\infty} (-1)^n \overline{\Upsilon}^{\overline{I}}_n \frac{1}{\zeta^n} \quad . \tag{3.14}$$

The action (3.13) is invariant under the global U(1) transformation

$$\Upsilon(\zeta) \quad \to \quad \Upsilon(e^{i\alpha}\zeta) \qquad \Longleftrightarrow \qquad \Upsilon_n \quad \to \quad e^{in\alpha}\Upsilon_n \quad . \tag{3.15}$$

[4] We use the relations $\Delta^\alpha = 2D^\alpha - \frac{1}{\zeta}\nabla^\alpha$, $\overline{\Delta}^{\dot\alpha} = -\frac{2}{\zeta}\overline{D}^{\dot\alpha} + \frac{1}{\zeta}\overline{\nabla}^{\dot\alpha}$ which imply that $\Delta^4 = -16\frac{1}{\zeta^2}D^2\overline{D}^2$ when it acts on projective superfields and is integrated on the 6D space-time coordinates.

Due to the truncation of the series, the $\mathcal{N}=1$ constraints on the component superfields Υ_n^I, $\overline{\Upsilon}_n^{\overline{I}}$ are

$$\overline{D}_{\dot\alpha}\Upsilon_0^I=0 \quad , \quad \overline{D}^2\Upsilon_1^I=\partial\,\Upsilon_0^I \qquad ; \qquad D_\alpha\overline{\Upsilon}_0^{\overline{I}}=0 \quad , \quad D^2\overline{\Upsilon}_1^{\overline{I}}=\overline{\partial}\,\overline{\Upsilon}_0^{\overline{I}} \quad , \tag{3.16}$$

with Υ_n^I, $\overline{\Upsilon}_n^{\overline{I}}$ $(n>1)$ unconstrained $\mathcal{N}=1$ superfields. The constraints (3.16) define a set of 6D chiral–nonminimal (CNM) hypermultiplets [7] given by $\Upsilon_0^I=\Phi^I$ and $\Upsilon_1^I=\Sigma^I$ extended with an infinite number of auxiliary superfields.

We observe that the action (3.13) has the same properties of the 4D, $\mathcal{N}=1$ case (3.12). It is invariant under Kähler transformations

$$K(\Upsilon,\breve{\Upsilon}) \longrightarrow K(\Upsilon,\breve{\Upsilon})+\Lambda(\Upsilon)+\overline{\Lambda}(\breve{\Upsilon}) \quad , \tag{3.17}$$

and holomorphic reparametrizations of the Kähler manifold $\Upsilon^I \quad \longrightarrow \quad f^I(\Upsilon^J)$.

The physical superfields

$$\Upsilon^I(\zeta)\Big|_{\zeta=0} \;=\; \Phi^I \qquad , \qquad \frac{\mathrm{d}\Upsilon^I(\zeta)}{\mathrm{d}\zeta}\Big|_{\zeta=0} \;=\; \Sigma^I \quad , \tag{3.18}$$

of the 6D CNM hypermultiplet can be regarded as parameters of the tangent bundle $T\mathcal{M}$ of the Kähler manifold $\mathcal{M}$.

The simplest example concerns a flat one–dimensional manifold with $K=\overline{\Phi}\Phi$ in (3.12). In this case the action (3.13) becomes

$$\int d^6x d^4\theta \left\{ \oint_C \frac{d\zeta}{2\pi i\zeta}\,\breve{\Upsilon}\Upsilon \right\} = \int d^6x d^4\theta \left\{ \overline{\Phi}\Phi - \overline{\Sigma}\Sigma + \sum_{n=2}^{+\infty}(-1)^n\overline{\Upsilon}_n\Upsilon_n \right\} \quad . \tag{3.19}$$

After integrating out the auxiliary superfields Υ_n, $\overline{\Upsilon}_n$ with $n>1$, we have $\int d^6x d^4\theta[\overline{\Phi}\Phi - \overline{\Sigma}\Sigma]$ which is the action for a free 6D $\mathcal{N}=(1,0)$ CNM hypermultiplet which has been investigated in [7]. In particular, it is dual to the free CC formulation (2.1).

The analysis of the free system can be extended to the non–trivial cases (3.13). We need eliminate the auxiliary superfields of the polar hypermultiplet. This can be done exactly as in the 4D case [13] where we refer the reader for details (see also [31] for recent applications). The action we are left with has the following form

$$\begin{aligned} S_{CNM}(\Phi^I,\overline{\Phi}^{\overline{I}},\Sigma^I,\overline{\Sigma}^{\overline{I}}) = \int d^6x d^4\theta \Big\{ & K(\Phi,\overline{\Phi}) - g_{I\overline{J}}(\Phi,\overline{\Phi})\Sigma^I\overline{\Sigma}^{\overline{J}} \\ & + \sum_{p=2}^{+\infty}\mathcal{R}_{I_1\cdots I_p\overline{J}_1\cdots\overline{J}_p}(\Phi,\overline{\Phi})\Sigma^{I_1}\ldots\Sigma^{I_p}\overline{\Sigma}^{\overline{J}_1}\ldots\overline{\Sigma}^{\overline{J}_p} \Big\} \quad , \end{aligned} \tag{3.20}$$

where the tensors $\mathcal{R}_{I_1\cdots I_p\overline{J}_1\cdots\overline{J}_p}$ are functions of the Riemann curvature $R_{I\overline{J}K\overline{L}}$ and its covariant derivatives. All the terms contain equal powers of Σ and $\overline{\Sigma}$ as a consequence of the invariance under (3.15). It is worth the mention, that presently, there is in general not known a closed-form analytic expression for $\mathcal{R}_{I_1\cdots I_p\overline{J}_1\cdots\overline{J}_p}(\Phi,\overline{\Phi})$. A solution to this problem would represent a major advance in understanding this class of problems.

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The action (3.20) describes a class of non–trivial 6D CNM sigma–models which are guaranteed to be on–shell $\mathcal{N} = (1,0)$ supersymmetric and 6D Lorentz invariant by construction.

So far we have restricted our attention to the polar multiplet as an extension of the 4D chiral multiplet. In particular, we have constructed 6D, $\mathcal{N} = (1,0)$ supersymmetric sigma–models defined over the tangent bundle $T\mathcal{M}$ of a Kähler manifold $\mathcal{M}$. In the four dimensional case, using the projective superspace, in [13, 14] an extension of the rigid c–map [32] was proposed which allows us to obtain a 4D, $\mathcal{N} = 2$ hyper-Kähler manifold starting from a 4D special Kähler geometry. The construction makes use of $O(2n)$ multiplets in projective superspace. Without giving any detail, we note that, as follows from our previous discussion, the construction of $O(2n)$ 6D, $\mathcal{N} = (1,0)$ hyper-Kähler sigma models along the lines of [13] should work straightforwardly since the dimensions of the space–time should not affect the superspace structures which allow for that construction.

We conclude by noting that our previous analysis covers only a small set of projective superspace sigma–models. The relevant property of actions of the form (3.13) is that the auxiliary superfields integration procedure is quite well understood and solved exactly for some non–trivial examples [13, 31]. It is believed that all the hyper-Kähler metrics can be derived from the most general polar multiplet action $K(\Upsilon, \breve{\Upsilon}, \zeta)$ with a non–trivial dependence on ζ [5]. We expect that the CNM's would arise naturally also in the general projective superspace case and the 6D structure would be the same as in our present analysis.

4. 6D, $\mathcal{N} = (1,0)$ CNM sigma-models

Six–dimensional projective superspace provides a powerful method to build a class of 6D, $\mathcal{N} = (1,0)$ supersymmetric nonlinear sigma–models whose partially on–shell description is given in terms of CNM 4D, $\mathcal{N} = 1$ superfields. The projective superspace construction insures that the resulting CNM sigma–model is on–shell 6D, $\mathcal{N} = (1,0)$ supersymmetric and we expect the structure of the CNM target space geometry to arise naturally. However, the action (3.20) for a 6D sigma–model as coming from projective superspace is not the most general action consistent with the symmetries of the problem.

In this section we investigate the most general class of CNM sigma–models we can construct directly in terms of 4D superfields and figure out the associated target space geometry, as done in section 2 for the CC case. In particular, we study how the defining tensors of the model are constrained by the demand of on–shell 6D, $\mathcal{N} = (1,0)$ SUSY.

Generalizing the free $\mathcal{N} = (1,0)$ CNM action [7], we consider the following ansatz for the most general $(1,0)$ CNM sigma–model action, off–shell invariant under 4D SUSY and the $\mathrm{Sl}(2,\mathbb{C})\times\mathrm{U}(1)$ subgroup of the 6D Lorentz group

$$S = \int d^6x \left[\int d^4\theta \, G\Big(\Phi^a, \overline{\Phi}^{\overline{a}}, \Sigma^k, \overline{\Sigma}^{\overline{k}}\Big) + \int d^2\theta \, P_a\Big(\Phi^b\Big)\partial\Phi^a + \int d^2\overline{\theta} \, \overline{P}_{\bar{a}}\Big(\overline{\Phi}^{\overline{b}}\Big)\overline{\partial\Phi}^{\overline{a}} \right] \quad , \quad (4.1)$$

[5] We thank Martin Roček for electronic correspondence on this point and for informing us on a forthcoming proof of this claim [33]. In harmonic superspace it is known that all hyper-Kähler metrics can be found from the most general q^+ hypermultiplet action [27, 29].

where the superfields Φ^a, $\overline{\Phi}^{\overline{a}}$, Σ^k, $\overline{\Sigma}^{\overline{k}}$ are CNM satisfying

$$\begin{aligned} \overline{D}_{\dot\alpha}\Phi^a &= 0 \quad , \quad \overline{D}^2\Sigma^k = S^k_a(\Phi)\,\partial\,\Phi^a \quad , \\ D_\alpha\overline{\Phi}^{\overline{a}} &= 0 \quad , \quad D^2\overline{\Sigma}^{\overline{k}} = \overline{S}^{\overline{k}}_{\overline{a}}(\overline{\Phi})\,\overline{\partial}\,\overline{\Phi}^{\overline{a}} \quad . \end{aligned} \tag{4.2}$$

The CNM models emerging from projective superspace correspond to the particular choice $P_a = 0$, $S = 1$ and G constrained to have the form (3.20).

In trying to keep the discussion very general we allow the number of chiral (n_c) and nonminimal (n_{nm}) superfields to be different, we generalize the nonminimal constraint by the introduction of the tensor $S^a_k(\Phi)$ and add holomorphic terms admitted by the symmetries of the theory.

Actually, the introduction of a holomorphic term is necessary whenever $n_{nm} < n_c$. As a particular example we mention the case of one free CC plus one free CNM pairs ($n_{nm} = 1$, $n_c = 3$)

$$S = \int d^6x \left[\int d^4\theta\,[\overline{\Phi}_+\Phi_+ + \overline{\Phi}_-\Phi_- + \overline{\Phi}\Phi - \overline{\Sigma}\Sigma] + \int d^2\theta\,\Phi_+\partial\Phi_- + \int d^2\overline{\theta}\,\overline{\Phi}_+\overline{\partial}\overline{\Phi}_- \right] , \tag{4.3}$$

$$\overline{D}_{\dot\alpha}\Phi_\pm = \overline{D}_{\dot\alpha}\Phi = 0, \quad , \quad D_\alpha\overline{\Phi}_\pm = D_\alpha\overline{\Phi} = 0 \quad , \quad \overline{D}^2\Sigma = \partial\,\Phi \quad , \quad D^2\overline{\Sigma} = \overline{\partial}\,\overline{\Phi} \quad . \tag{4.4}$$

While the completion to 6D of the CNM (Φ, Σ) kinetic terms is provided by the non–trivial constraint $\overline{D}^2\Sigma = \partial\Phi$, the completion of the kinetic terms for $\Phi_\pm$ makes use of the holomorphic term, as discussed in [7] and in section 2.

Now, we go back to the general case (4.1, 4.2). In the CC case of section 2 we have first imposed the restoration of 6D Lorentz invariance on the bosonic part of the action with the auxiliary fields set on–shell. The requirement of 6D Lorentz invariance constrains the target space to be hyper-Kähler and this is sufficient to guarantee the on–shell invariance of the whole action plus 6D, $\mathcal{N} = (1, 0)$ supersymmetry. We now follow the same approach to constrain the tensors G, P, S, $\overline{P}$ and $\overline{S}$ of the CNM sigma–models (4.1, 4.2).

Having defined the component fields as in (A.1), we reduce the action (4.1) in components. The resulting action is much more complicated than the CC one and we refer the reader to appendix B for the whole component lagrangian (see eq. (B.1)).

Before performing the auxiliary fields integration, it is useful to write the bosonic part of (B.1) in a compact form introducing a vectorial/matricial notation. We define the following matrices ($G_a \equiv \frac{\partial G}{\partial \Phi^a}$)

$$M \equiv \begin{pmatrix} 0 & G_{a\overline{b}} \\ G_{\overline{a}b} & 0 \end{pmatrix} \quad , \quad N \equiv \begin{pmatrix} G_{ar} & G_{a\overline{r}} \\ G_{\overline{a}r} & G_{\overline{a}\overline{r}} \end{pmatrix} \quad , \quad H \equiv \begin{pmatrix} G_{kr} & G_{k\overline{r}} \\ G_{\overline{k}r} & G_{\overline{k}\overline{r}} \end{pmatrix} \quad , \tag{4.5}$$

$$\mathcal{S} \equiv \begin{pmatrix} S^k_b & 0 \\ 0 & \overline{S}^{\overline{k}}_{\overline{b}} \end{pmatrix} \quad , \quad P \equiv \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \quad , \quad P_\pm \equiv \frac{1}{2}(1 \pm P) \quad , \tag{4.6}$$

$$\mathcal{O} \equiv \begin{pmatrix} (G_k S^k_b + P_b)_{(a)} - (G_k S^k_a + P_a)_{(b)} & -S^k_a G_{k\overline{b}} \\ -\overline{S}^{\overline{k}}_{\overline{a}} G_{\overline{k}b} & (G_{\overline{k}}\overline{S}^{\overline{k}}_{\overline{b}} + \overline{P}_{\overline{b}})_{(\overline{a})} - (G_{\overline{k}}\overline{S}^{\overline{k}}_{\overline{a}} + \overline{P}_{\overline{a}})_{(\overline{b})} \end{pmatrix} \quad . \tag{4.7}$$

in terms of which the bosonic component lagrangian becomes

$$
\begin{aligned}
S_{0f} = \int d^6x \Big[& -\frac{1}{4}\,\partial^{\alpha\dot\alpha}\mathcal{A}^T\, M\, \partial_{\alpha\dot\alpha}\mathcal{A} + \frac{1}{8}\,\partial^{\alpha\dot\alpha}\mathcal{B}^T\,[3H + PHP]\,\partial_{\alpha\dot\alpha}\mathcal{B} \\
& + \frac{1}{8}\,\partial^{\alpha\dot\alpha}\mathcal{A}^T\,[P,[P,N]]\,\partial_{\alpha\dot\alpha}\mathcal{B} + \partial\mathcal{A}^T\, P_+\mathcal{S}^T H\mathcal{S}P_-\,\overline{\partial}\mathcal{A} \\
& + \frac{1}{2}\,\mathcal{F}^T\, M\,\mathcal{F} + \frac{1}{8}\,\mathcal{H}^T\,[P,[P,H]]\,\mathcal{H} + \frac{1}{4}\,\mathcal{F}^T\,[P,[P,N]]\,\mathcal{H} \\
& + \mathcal{F}^T\, P_+\mathcal{O}\,\partial\mathcal{A} + \mathcal{F}^T\, P_-\mathcal{O}\,\overline{\partial}\mathcal{A} + \mathcal{H}^T\, P_+ H\mathcal{S}P_+\,\partial\mathcal{A} \\
& + \mathcal{H}^T\, P_- H\mathcal{S}P_-\,\overline{\partial}\mathcal{A} - \mathcal{F}^T\, P_+\mathcal{S}^T H\,\partial\mathcal{B} - \mathcal{F}^T\, P_-\mathcal{S}^T H\,\overline{\partial}\mathcal{B} \\
& - \frac{1}{2}\,\mathcal{P}^T_{\alpha\dot\alpha}\, H\,\mathcal{P}^{\alpha\dot\alpha} + \frac{1}{2}\,\mathcal{P}^T_{\alpha\dot\alpha}\,[P,N^T]\, i\partial^{\alpha\dot\alpha}\mathcal{A} + \frac{1}{2}\,\mathcal{P}^T_{\alpha\dot\alpha}\,\{P,H\}\, i\partial^{\alpha\dot\alpha}\mathcal{B}\Big] \quad .
\end{aligned}
\tag{4.8}
$$

As usual, the equations of motion for the auxiliary fields are algebraic. Defining the matrices

$$
\widetilde{P} \equiv \left(\begin{array}{c|c} P & 0 \\ \hline 0 & P \end{array}\right) \quad , \quad \mathcal{G} \equiv \left(\begin{array}{c|c} M & N \\ \hline N^T & H \end{array}\right) \quad , \tag{4.9}
$$

$$
\mathcal{Z} = \left(\tfrac{1}{4}[\widetilde{P},[\widetilde{P},\mathcal{G}]]\right)^{-1} \quad , \quad \mathcal{X}_\pm = \tfrac{1}{2}(1 \pm \widetilde{P}) \left(\begin{array}{c|c} \mathcal{O} & -\mathcal{S}^T H \\ \hline H\mathcal{S}P_\pm & 0 \end{array}\right) \quad , \tag{4.10}
$$

the solution to the equations of motion for the auxiliary fields read

$$
\begin{aligned}
\begin{pmatrix} \mathcal{F} \\ \mathcal{H} \end{pmatrix} &= -\,\mathcal{Z}\mathcal{X}_+\,\partial \begin{pmatrix} \mathcal{A} \\ \mathcal{B} \end{pmatrix} - \mathcal{Z}\mathcal{X}_-\,\overline{\partial} \begin{pmatrix} \mathcal{A} \\ \mathcal{B} \end{pmatrix} \quad , \\
\mathcal{P}_{\alpha\dot\alpha} &= \frac{1}{2}\, H^{-1}[P,N^T]\, i\partial_{\alpha\dot\alpha}\mathcal{A} + \frac{1}{2}\, H^{-1}\{P,H\}\, i\partial_{\alpha\dot\alpha}\mathcal{B} \quad .
\end{aligned}
\tag{4.11}
$$

Inserting back into (4.8) and defining

$$
\mathcal{C} \equiv \begin{pmatrix} \mathcal{A} \\ \mathcal{B} \end{pmatrix} \quad , \quad \mathcal{Y} \equiv \mathcal{X}_+^T\mathcal{Z}\mathcal{X}_- - \left(\begin{array}{c|c} P_+\mathcal{S}^T H\mathcal{S}P_- & 0 \\ \hline 0 & 0 \end{array}\right) \quad , \tag{4.12}
$$

$$
\mathcal{K} \equiv \left(\begin{array}{c|c} \mathcal{K}_1 & \mathcal{K}_2 \\ \hline \mathcal{K}_2^T & \mathcal{K}_3 \end{array}\right) \quad , \quad \mathcal{K}_1 = M + \frac{1}{2}[P,N]H^{-1}[N^T,P] \quad , \tag{4.13}
$$

$$
\mathcal{K}_2 = \frac{1}{2}[N,P]H^{-1}\{P,H\} - \frac{1}{4}[P,[P,N]] \quad , \quad \mathcal{K}_3 = \frac{1}{2}HPH^{-1}PH - \frac{1}{2}H \quad , \tag{4.14}
$$

we find the following action for the bosonic physical fields

$$
S = -\int d^6x \left[\frac{1}{4}\partial^{\alpha\dot\alpha}\mathcal{C}^T\,\mathcal{K}\,\partial_{\alpha\dot\alpha}\mathcal{C} + \partial\mathcal{C}^T\,\mathcal{Y}\,\overline{\partial}\mathcal{C} \right] \quad . \tag{4.15}
$$

The matrix $\mathcal{Y}$ defined in (4.12) is not symmetric. In order to proceed we need symmetrize it. To this porpose we rewrite (4.15) as

$$
-\frac{1}{4}\int d^6x \left[\partial^{\alpha\dot\alpha}\mathcal{C}^T\;\mathcal{K}\;\partial_{\alpha\dot\alpha}\mathcal{C} \;+\; \partial\mathcal{C}^T\;\widetilde{\mathcal{K}}\;\overline{\partial}\mathcal{C} \;+\; \overline{\partial}\mathcal{C}^T\;\widetilde{\mathcal{K}}\;\partial\mathcal{C} \right]
$$

$$+\,\frac{1}{2}\int d^6x\,\partial\mathcal{C}^T(\mathcal{Y}-\mathcal{Y}^T)\,\overline{\partial}\mathcal{C} \quad , \tag{4.16}$$

where we have defined

$$\widetilde{\mathcal{K}}\equiv\mathcal{Y}+\mathcal{Y}^T \;=\; \widetilde{\mathcal{X}}^T\mathcal{Z}\widetilde{\mathcal{X}}\;-\;\left(\begin{array}{c|c}\frac{1}{4}[P,[P,\mathcal{S}^THS]] & 0\\ \hline 0 & 0\end{array}\right) \quad , \tag{4.17}$$

and

$$\widetilde{\mathcal{X}} \;=\; \mathcal{X}_+ \,+\, \mathcal{X}_- \;=\; \left(\begin{array}{c|c}\mathcal{O} & -\mathcal{S}^TH\\ \hline P_+H\mathcal{S}P_+ \,+\, P_-H\mathcal{S}P_- & 0\end{array}\right) \quad . \tag{4.18}$$

Note that the structure of (4.16) is similar to the one for the CC case (see eq. (2.17)). Therefore, by imposing the restoration of 6D Lorentz invariance we obtain the following condition

$$\mathcal{K}=\widetilde{\mathcal{K}} \quad . \tag{4.19}$$

As in the CC case we expect the constraint (4.19) to be sufficient to make the second line of (4.16) a total derivative and, more importantly, to provide on–shell 6D Lorentz invariance and 6D, $\mathcal{N}=(1,0)$ SUSY of the whole action. Unfortunately, in this case the direct proof is not straightforward and we have not pursued the calculations up to the very end.

The constraint (4.19), once written for each component of the two matrices, gives rise to a system of equations which is much more intricated than (2.20) for the CC case. Up to now we have not been able to solve it in general. We are going to provide the explicit solution only in the following example.

Example: 4D target space. We consider the CNM sigma model describing the dynamics of one chiral and one nonminimal superfields defined by the action[6]

$$S=\int d^6xd^4\theta\,G(\Phi,\overline{\Phi},\Sigma,\overline{\Sigma}) \quad , \quad \overline{D}^2\Sigma \;=\; \partial\,\Phi \quad , \quad D^2\overline{\Sigma} \;=\; \overline{\partial}\,\overline{\Phi} \quad . \tag{4.20}$$

In this case we can write explicitly all the quantities which enter our equations (4.19). In particular, $\mathcal{K}$ (4.13, 4.14) has component matrices given by

$$\mathcal{K}_1 \;=\; \frac{1}{\det H}\begin{pmatrix} 2G^2_{\Phi\overline{\Sigma}}G_{\Sigma\Sigma} & G_{\Phi\overline{\Phi}}(\det H)+2G_{\Phi\overline{\Sigma}}G_{\Sigma\overline{\Phi}}G_{\Sigma\overline{\Sigma}} \\ G_{\Phi\overline{\Phi}}(\det H)+2G_{\Phi\overline{\Sigma}}G_{\Sigma\overline{\Phi}}G_{\Sigma\overline{\Sigma}} & 2G^2_{\Sigma\overline{\Phi}}G_{\overline{\Sigma}\overline{\Sigma}}\end{pmatrix} \quad , \tag{4.21}$$

$$\mathcal{K}_2 \;=\; \frac{1}{\det H}\begin{pmatrix} 2G_{\Phi\overline{\Sigma}}G_{\Sigma\Sigma}G_{\Sigma\overline{\Sigma}} & G_{\Phi\overline{\Sigma}}(G_{\Sigma\Sigma}G_{\overline{\Sigma}\overline{\Sigma}}+G^2_{\Sigma\overline{\Sigma}}) \\ G_{\Sigma\overline{\Phi}}(G_{\Sigma\Sigma}G_{\overline{\Sigma}\overline{\Sigma}}+G^2_{\Sigma\overline{\Sigma}}) & 2G_{\Sigma\overline{\Phi}}G_{\Sigma\overline{\Sigma}}G_{\overline{\Sigma}\overline{\Sigma}}\end{pmatrix} \quad , \tag{4.22}$$

$$\mathcal{K}_3 \;=\; \frac{1}{\det H}\begin{pmatrix} 2G_{\Sigma\Sigma}G^2_{\Sigma\overline{\Sigma}} & G_{\Sigma\overline{\Sigma}}(G_{\Sigma\Sigma}G_{\overline{\Sigma}\overline{\Sigma}}+G^2_{\Sigma\overline{\Sigma}}) \\ G_{\Sigma\overline{\Sigma}}(G_{\Sigma\Sigma}G_{\overline{\Sigma}\overline{\Sigma}}+G^2_{\Sigma\overline{\Sigma}}) & 2G^2_{\Sigma\overline{\Sigma}}G_{\overline{\Sigma}\overline{\Sigma}}\end{pmatrix} \quad , \tag{4.23}$$

[6] We consider the simplest CNM constraint with $S(\Phi)=1$ since, in the present case, we can always eliminate the function $S(\Phi)$ by a redefinition of the nonminimal superfield, $\Sigma\equiv S(\Phi)\Sigma'$, which implies $\overline{D}^2\Sigma'=\partial\Phi$.

where $(\det H) = (G_{\Sigma\Sigma}G_{\overline{\Sigma\Sigma}} - G^2_{\Sigma\overline{\Sigma}})$. Furthermore, we have

$$\mathcal{Z} = \frac{1}{G_{\Phi\overline{\Phi}}G_{\Sigma\overline{\Sigma}} - G_{\Phi\overline{\Sigma}}G_{\Sigma\overline{\Phi}}} \left(\begin{array}{cc|cc} 0 & G_{\Sigma\overline{\Sigma}} & 0 & -G_{\Sigma\overline{\Phi}} \\ G_{\Sigma\overline{\Sigma}} & 0 & -G_{\Phi\overline{\Sigma}} & 0 \\ \hline 0 & -G_{\Phi\overline{\Sigma}} & 0 & G_{\Phi\overline{\Phi}} \\ -G_{\Sigma\overline{\Phi}} & 0 & G_{\Phi\overline{\Phi}} & 0 \end{array} \right) \quad , \tag{4.24}$$

and $\widetilde{\mathcal{K}} = \left(\begin{array}{c|c} \widetilde{\mathcal{K}}_1 & \widetilde{\mathcal{K}}_2 \\ \hline \widetilde{\mathcal{K}}_2^T & \widetilde{\mathcal{K}}_3 \end{array} \right)$ in (4.17) becomes ($\tilde{k} \equiv (G_{\Phi\overline{\Phi}}G_{\Sigma\overline{\Sigma}} - G_{\Phi\overline{\Sigma}}G_{\Sigma\overline{\Phi}})$)

$$\widetilde{\mathcal{K}}_1 = \frac{1}{\tilde{k}} \begin{pmatrix} 2G^2_{\Phi\overline{\Sigma}}G_{\Sigma\Sigma} & G_{\Phi\overline{\Phi}}G_{\Sigma\Sigma}G_{\overline{\Sigma\Sigma}} + G_{\Phi\overline{\Sigma}}G_{\Sigma\overline{\Phi}}G_{\Sigma\overline{\Sigma}} \\ G_{\Phi\overline{\Phi}}G_{\Sigma\Sigma}G_{\overline{\Sigma\Sigma}} + G_{\Phi\overline{\Sigma}}G_{\Sigma\overline{\Phi}}G_{\Sigma\overline{\Sigma}} & 2G^2_{\Sigma\overline{\Phi}}G_{\overline{\Sigma\Sigma}} \end{pmatrix} \quad , \tag{4.25}$$

$$\widetilde{\mathcal{K}}_2 = \frac{1}{\tilde{k}} \begin{pmatrix} 2G_{\Phi\overline{\Sigma}}G_{\Sigma\Sigma}G_{\Sigma\overline{\Sigma}} & G_{\Phi\overline{\Sigma}}(G_{\Sigma\Sigma}G_{\overline{\Sigma\Sigma}} + G^2_{\Sigma\overline{\Sigma}}) \\ G_{\Sigma\overline{\Phi}}(G_{\Sigma\Sigma}G_{\overline{\Sigma\Sigma}} + G^2_{\Sigma\overline{\Sigma}}) & 2G_{\Sigma\overline{\Phi}}G_{\Sigma\overline{\Sigma}}G_{\overline{\Sigma\Sigma}} \end{pmatrix} \quad , \tag{4.26}$$

$$\widetilde{\mathcal{K}}_3 = \frac{1}{\tilde{k}} \begin{pmatrix} 2G_{\Sigma\Sigma}G^2_{\Sigma\overline{\Sigma}} & G_{\Sigma\overline{\Sigma}}(G_{\Sigma\Sigma}G_{\overline{\Sigma\Sigma}} + G^2_{\Sigma\overline{\Sigma}}) \\ G_{\Sigma\overline{\Sigma}}(G_{\Sigma\Sigma}G_{\overline{\Sigma\Sigma}} + G^2_{\Sigma\overline{\Sigma}}) & 2G^2_{\Sigma\overline{\Sigma}}G_{\overline{\Sigma\Sigma}} \end{pmatrix} \quad , \tag{4.27}$$

Now, imposing $\mathcal{K} = \widetilde{\mathcal{K}}$ as in (4.19) the only non–trivial condition we obtain is

$$G_{\Sigma\Sigma}G_{\overline{\Sigma\Sigma}} - G^2_{\Sigma\overline{\Sigma}} = G_{\Phi\overline{\Phi}}G_{\Sigma\overline{\Sigma}} - G_{\Phi\overline{\Sigma}}G_{\Sigma\overline{\Phi}} \quad . \tag{4.28}$$

In this case one can check that this condition is sufficient for the second line of (4.16) to be a total derivative. As we are going to show at the end of section 6 the condition (4.28) implies that the 4D target space geometry is hyper-Kähler.

It is interesting to note that (4.28) is exactly the same constraint which was found in [19] from the condition of vanishing one–loop beta–function for a 2D CNM sigma–model with $\mathcal{N} = 4$ supersymmetry. This implies that the resulting manifold is Ricci-flat and being four dimensional, it is necessarily hyper-Kähler [34, 19].

5. Duality between 6D, $\mathcal{N} = (1, 0)$ CC and CNM sigma-models

One of the very interesting properties of the nonminimal superfield in four, and lower dimensions, is that it is dual to the chiral multiplet [20]. In [7] we proved that, in flat target spaces, an analogous duality exists between 6D, $\mathcal{N} = (1, 0)$ CC and CNM hypermultiplets. The same happens for 5D sigma–models [14] . In this section we address the issue of duality for 6D, $\mathcal{N} = (1, 0)$ nonlinear sigma–models.

We start by considering the most general CNM sigma–model (4.1). To build its dual we implement the CNM constraint (4.2) using a lagrangian multiplier. We then consider the action

$$\begin{aligned} S = & \int d^6x d^4\theta \, G\Big(\Phi^a, \overline{\Phi}^{\overline{a}}, \Sigma^k, \overline{\Sigma}^{\overline{k}}\Big) + \int d^6x d^2\theta \, P_a\Big(\Phi^b\Big) \partial \, \Phi^a + \int d^6x d^2\overline{\theta} \, \overline{P}_{\overline{a}}\Big(\overline{\Phi}^{\overline{b}}\Big) \overline{\partial} \, \overline{\Phi}^{\overline{a}} \\ & - \int d^6x d^4\theta \left[Y_k\Big(\overline{D}^2\Sigma^k - S^k_a(\Phi) \, \partial \, \Phi^a\Big) + \overline{Y}_{\overline{k}}\Big(D^2\overline{\Sigma}^{\overline{k}} - \overline{S}^{\overline{k}}_{\overline{a}}(\Phi) \, \overline{\partial} \, \overline{\Phi}^{\overline{a}}\Big) \right] \quad , \end{aligned} \tag{5.1}$$

where Y_k, $\overline{Y}_{\overline{k}}$, Σ^k and $\overline{\Sigma}^{\overline{k}}$ are unconstrained complex superfields. Integrating out Y_k and $\overline{Y}_{\overline{k}}$ we are back to the original CNM model (4.1, 4.2). On the other hand, varying with respect to Σ^k and $\overline{\Sigma}^{\overline{k}}$ we obtain the equations of motion ($G_k \equiv \frac{\partial G}{\partial \Sigma^k}$)

$$G_k = \overline{D}^2 Y_k \quad , \quad G_{\overline{k}} = D^2 \overline{Y}_{\overline{k}} \quad . \tag{5.2}$$

We can integrate out Σ^k and $\overline{\Sigma}^{\overline{k}}$ defining new (anti)chiral superfields $\chi_k \equiv \overline{D}^2 Y_k$, $\overline{\chi}_{\overline{k}} \equiv D^2 \overline{Y}_{\overline{k}}$ and inverting the equations (5.2)

$$G_k\Big(\Phi^a, \overline{\Phi}^{\overline{a}}, \Sigma^k, \overline{\Sigma}^{\overline{k}}\Big) = \chi_k \implies \Sigma^k = \Sigma^k\Big(\Phi^a, \overline{\Phi}^{\overline{a}}, \chi_k, \overline{\chi}_{\overline{k}}\Big) \quad , \tag{5.3}$$

$$G_{\overline{k}}\Big(\Phi^a, \overline{\Phi}^{\overline{a}}, \Sigma^k, \overline{\Sigma}^{\overline{k}}\Big) = \overline{\chi}_{\overline{k}} \implies \overline{\Sigma}^{\overline{k}} = \overline{\Sigma}^{\overline{k}}\Big(\Phi^a, \overline{\Phi}^{\overline{a}}, \chi_k, \overline{\chi}_{\overline{k}}\Big) \quad . \tag{5.4}$$

Substituting back into (5.1) we find the action for the dual CC model

$$\begin{aligned} &\int d^6x \Bigg\{ \int d^4\theta \Bigg[G\Big(\Phi^a, \overline{\Phi}^{\overline{a}}, \Sigma^k, \overline{\Sigma}^{\overline{k}}\Big) - \Sigma^k \chi_k - \overline{\Sigma}^{\overline{k}} \overline{\chi}_{\overline{k}} \Bigg]\Bigg|_{\Sigma^k = \Sigma^k(\Phi^a, \overline{\Phi}^{\overline{a}}, \chi_k, \overline{\chi}_{\overline{k}})} \\ &\qquad + \int d^2\theta \Bigg[\Big(\chi_k S^k_a + P_a\Big) \partial\, \Phi^a \Bigg] + \int d^2\overline{\theta} \Bigg[\Big(\overline{\chi}_{\overline{k}} \overline{S}^{\overline{k}}_{\overline{a}} + \overline{P}_{\overline{a}}\Big) \overline{\partial}\, \overline{\Phi}^{\overline{a}} \Bigg] \Bigg\} \\ &\equiv \int d^6x \Bigg\{ \int d^4\theta\, \widetilde{G}\Big(\Psi^I, \overline{\Psi}^{\overline{I}}\Big) + \int d^2\theta\, Q_I\Big(\Psi^J\Big) \partial\, \Psi^I + \int d^2\overline{\theta}\, \overline{Q}_{\overline{I}}\Big(\overline{\Psi}^{\overline{J}}\Big) \overline{\partial}\, \overline{\Psi}^{\overline{I}} \Bigg\} \quad . \end{aligned} \tag{5.5}$$

where we have defined the (anti)chiral superfields ($\overline{\Psi}^{\overline{I}}$) Ψ^I

$$\Psi^I = \begin{pmatrix} \Phi^a \\ \chi_k \end{pmatrix} \quad , \quad \overline{\Psi}^{\overline{I}} = \begin{pmatrix} \overline{\Phi}^{\overline{a}} \\ \overline{\chi}_{\overline{k}} \end{pmatrix} \quad , \tag{5.6}$$

being the coordinates of the dual target space. The Kähler potential $\widetilde{G}$ is the Legendre transform of G in (4.1) and the (anti)holomorphic pieces are expressed in terms of

$$Q_I \equiv \begin{pmatrix} \chi_k S^k_a + P_a \\ 0 \end{pmatrix} \quad , \quad \overline{Q}_{\overline{I}} \equiv \begin{pmatrix} \overline{\chi}_{\overline{k}} \overline{S}^{\overline{k}}_{\overline{a}} + \overline{P}_{\overline{a}} \\ 0 \end{pmatrix} \quad . \tag{5.7}$$

This procedure is very general and allows to map any CNM sigma–model (4.1, 4.2) to a CC sigma–model (5.5–5.7).

A very interesting subclass of dual CC–CNM pairs are those coming from projective superspace. Using the prescription just described, given the CNM sigma–model (3.20) we can find the corresponding on–shell 6D, $\mathcal{N} = (1,0)$ CC sigma–model. Since in the projective case of section 3 the CNM multiplet is naturally interpreted as parametrizing the tangent bundle $T\mathcal{M}$ of the Kähler manifold $\mathcal{M}$, once we perform a duality transformation, the resulting CC coordinates $(\Phi^a, \overline{\Phi}^{\overline{a}}, \chi_a, \overline{\chi}_{\overline{a}})$ describe the cotangent bundle $T^*\mathcal{M}$ of $\mathcal{M}$. These manifolds must be hyper-Kähler as requested from the general analysis of the CC case (see section 2).

It is important to note that in the projective case as well in the free case, the holomorphic term has the particular form ($P = 0$, $S = 1$)

$$\int d^2\theta\,\chi_a \partial\,\Phi^a \;=\; \frac{1}{2}\int d^2\theta\,\left(\Psi^J \Omega^0_{JI}\right)\partial\Psi^I \quad , \tag{5.8}$$

where

$$\Omega^0_{IJ} = \begin{pmatrix} 0 & -\delta^r_a \\ \delta^b_k & 0 \end{pmatrix} \quad , \tag{5.9}$$

is the constant symplectic matrix.

The holomorphic term appearing in (5.5) from the CNM→CC dualization is at most linear in the dualized (anti)chiral superfield ($\overline{\chi}_{\overline{k}}$) χ_k. At a first sight this term seems to describe only a subclass of models and one may wonder whether the duality map does indeed generate the entire class of CC sigma–models. To answer this question we now prove that performing a suitable change of coordinates, *any* holomorphic term in (2.5) can be always reduced locally to the canonical form (5.8). Therefore, we can state that the duality map described above is the most general one and relates the whole class of CNM models (4.1, 4.2) to the whole class of CC models (2.5).

To this end we consider the most general CC sigma–model (2.5) and search for a holomorphic change of coordinates

$$\Psi'^a(\Psi) = f^a(\Psi)\ ,\ \overline{\Psi}'^{\overline{a}}(\overline{\Psi}) = \overline{f}^{\overline{a}}(\overline{\Psi}) \quad , \tag{5.10}$$

$$\Psi^a(\Psi') = (f^{-1})^a(\Psi')\ ,\ \overline{\Psi}^{\overline{a}}(\overline{\Psi}') = (\overline{f}^{-1})^{\overline{a}}(\overline{\Psi}') \quad , \tag{5.11}$$

such that

$$Q_a(\Psi(\Psi'))\,\partial\,\Psi^a(\Psi') \equiv \frac{1}{2}\Psi'^b\,\Omega^0_{ba}\,\partial\,\Psi'^a + \frac{\partial g(\Psi')}{\partial\Psi'^a}\,\partial\,\Psi'^a \quad , \tag{5.12}$$

$$\overline{Q}_{\overline{a}}(\overline{\Psi}(\overline{\Psi}'))\,\overline{\partial}\,\overline{\Psi}^{\overline{a}}(\overline{\Psi}') \equiv \frac{1}{2}\overline{\Psi}'^{\overline{b}}\,\overline{\Omega}^0_{\overline{b}\overline{a}}\,\overline{\partial}\,\overline{\Psi}'^{\overline{a}} + \frac{\partial\overline{g}(\overline{\Psi}')}{\partial\overline{\Psi}'^{\overline{a}}}\,\overline{\partial}\,\overline{\Psi}'^{\overline{a}} \quad . \tag{5.13}$$

The terms $\frac{\partial g(\Psi')}{\partial\Psi'^a}\,\partial\,\Psi'^a = \partial\,g(\Psi')$ and $\frac{\partial\overline{g}(\overline{\Psi}')}{\partial\overline{\Psi}'^{\overline{a}}}\,\overline{\partial}\,\overline{\Psi}'^{\overline{a}} = \overline{\partial}\,\overline{g}(\overline{\Psi}')$, being total derivatives, do not affect the holomorphic term and can be always admitted in a change of coordinates.

The previous equations are equivalent to the following differential equations for the functions f^a and $\overline{f}^{\overline{a}}$

$$\frac{1}{2}\,f^c\,\Omega^0_{cb}\,\frac{\partial f^b}{\partial\Psi^a} + \frac{\partial g}{\partial\Psi^a} = Q_a \quad , \tag{5.14}$$

$$\frac{1}{2}\,\overline{f}^{\overline{c}}\,\overline{\Omega}^0_{\overline{c}\overline{b}}\,\frac{\partial\overline{f}^{\overline{b}}}{\partial\overline{\Psi}^{\overline{a}}} + \frac{\partial\overline{g}}{\partial\overline{\Psi}^{\overline{a}}} = \overline{Q}_{\overline{a}} \quad , \tag{5.15}$$

and also

$$\left(Q_{d(c)} - Q_{c(d)}\right)\frac{\partial\Psi^c}{\partial\Psi'^a}\frac{\partial\Psi^d}{\partial\Psi'^b} = \Omega_{cd}\,\frac{\partial\Psi^c}{\partial\Psi'^a}\frac{\partial\Psi^d}{\partial\Psi'^b} = \Omega^0_{ab} \quad , \tag{5.16}$$

$$\left(\overline{Q}_{\overline{d}(\overline{c})} - \overline{Q}_{\overline{c}(\overline{d})}\right)\frac{\partial\overline{\Psi}^{\overline{c}}}{\partial\overline{\Psi}'^{\overline{a}}}\frac{\partial\overline{\Psi}^{\overline{d}}}{\partial\overline{\Psi}'^{\overline{b}}} = \overline{\Omega}_{\overline{c}\overline{d}}\,\frac{\partial\overline{\Psi}^{\overline{c}}}{\partial\overline{\Psi}'^{\overline{a}}}\frac{\partial\overline{\Psi}^{\overline{d}}}{\partial\overline{\Psi}'^{\overline{b}}} = \overline{\Omega}^0_{\overline{a}\overline{b}} \quad , \tag{5.17}$$

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where the components of the holomorphic two–form Ω and $\overline{\Omega}$ (2.19) of the hyper-Kähler manifold appear. The functions Q_a transform as the components of a holomorphic one–form $Q = Q_a \, d\Psi^a$ and the local change of coordinates described by the previous equations is such that the closed, nondegenerate, covariantly constant two–form $\Omega = -\partial \, Q$ is mapped to the canonical constant symplectic two–form Ω^0. The hyper-Kähler manifold is a complex symplectic manifold with respect to the holomorphic two–form Ω. According to Darboux theorem we can always choose a particular system of coordinates[7] [35, 23, 36] for which $\Omega = \Omega^0$ and $\overline{\Omega} = \overline{\Omega}^0$. This insures that locally our previous equations are always soluble.

The discussion above means that locally the 6D CC sigma–models can be always described, after the appropriate change of coordinates, by a holomorphic term having the canonical form (5.8). In the symplectic coordinates it is natural to divide the coordinates as in (5.6) and all the CC hyper-Kähler sigma–models in the symplectic basis reduce to

$$S = \int d^6x d^4\theta \, K'(\Phi, \overline{\Phi}, \chi, \overline{\chi}) + \int d^6x d^2\theta \, \chi_I \, \partial \, \Phi^I \, + \int d^6x d^2\overline{\theta} \, \overline{\chi}_{\overline{I}} \, \overline{\partial} \, \overline{\Phi}^{\overline{I}} \quad , \tag{5.18}$$

with K' the Kähler potential in the symplectic basis.

So far we have described how to dualize CNM models obtaining CC sigma–models. Now, we want to proceed in the other way around and construct the CNM dual of a general 6D, $\mathcal{N} = (1, 0)$ hyper-Kähler CC sigma–model. Once we have written it in Darboux coordinates as in (5.18), the CC$\rightarrow$CNM dualization goes straightforwardly.

We solve the kinematical (anti)chirality constraints of ($\overline{\chi}_{\overline{I}}$) χ_I in terms of an unconstrained complex superfield ($\overline{Y}_{\overline{I}}$) Y_I which plays a role similar to the Lagrange multiplier of (5.5)

$$\chi_I = -\overline{D}^2 Y_I \quad , \quad \overline{\chi}_{\overline{I}} = -D^2 \overline{Y}_{\overline{I}} \quad . \tag{5.19}$$

The action (5.18) takes the form

$$\int d^4\theta \left[K'(\Phi, \overline{\Phi}, \chi, \overline{\chi}) \, - \, Y_I \, \partial \, \Phi^I \, - \, \overline{Y}_{\overline{I}} \, \overline{\partial} \, \overline{\Phi}^{\overline{I}} \right] \quad . \tag{5.20}$$

Varying with respect to ($\overline{Y}_{\overline{I}}$) Y_I we obtain

$$\overline{D}^2 \, \frac{\partial K'}{\partial \chi_I} \, = \partial \, \Phi^I \quad , \quad D^2 \, \frac{\partial K'}{\partial \overline{\chi}_{\overline{I}}} \, = \overline{\partial} \, \overline{\Phi}^{\overline{I}} \quad . \tag{5.21}$$

Therefore, the superfields

$$\Sigma^I \, \equiv \, \frac{\partial K'}{\partial \chi_I} \quad , \quad \overline{\Sigma}^I \, \equiv \, \frac{\partial K'}{\partial \overline{\chi}_{\overline{I}}} \quad . \tag{5.22}$$

satisfy the linear constraints in (4.2) with $S = 1$. We invert these relations to determine χ_I and $\overline{\chi}_{\overline{I}}$ as functions of $(\Phi, \overline{\Phi}, \Sigma, \overline{\Sigma})$. Substituting back into the action (5.20) we find that

[7] See section five of [36] for an interesting discussion on Darboux coordinates in the case of generalized Kähler geometry. In their language, our (anti)holomorphic two-forms $\overline{\Omega}$ and Ω are those which define the inverse of a Poisson structure on the manifold.

the CC sigma–model (5.18) is dual to the CNM sigma–model defined by

$$\begin{aligned} S &= \int d^6x d^4\theta \left[K'\left(\Phi^I, \overline{\Phi}^{\overline{I}}, \chi_I, \overline{\chi}_{\overline{I}}\right) - \Sigma^I \chi_I - \overline{\Sigma}^{\overline{I}} \overline{\chi}_{\overline{I}} \right]\Bigg|_{\chi_I = \chi_I(\Phi^I, \overline{\Phi}^{\overline{I}}, \Sigma^I, \overline{\Sigma}^{\overline{I}})} \\ &= \int d^6x d^4\theta\, \widetilde{K}'\left(\Phi^I, \overline{\Phi}^{\overline{I}}, \Sigma^I, \overline{\Sigma}^{\overline{I}}\right) \quad , \end{aligned} \tag{5.23}$$

where $\widetilde{K}'$ is the Legendre transform with respect to $\overline{\chi}$ and χ of the Kähler potential K'. We have then found that all the CC sigma–models of section 2 written in a canonical symplectic system of coordinates are dual to CNM models.

So far we have considered maximal duality maps, i.e. trasformations where *all* the nonminimal multiplets are dualized to chirals and viceversa. However, one can consider more general situations where the duality map involves only a subset of superfields. These partial dualizations can be used to map a CNM model with $n_c \neq n_{nm}$ to a model with $n_c = n_{nm}$. This is possible every time $n_c - n_{nm} = 2n$.

On–shell pairs of dual CC–CNM sigma–models have the same dynamics. This means that the target space described by the two sigma–models is the same. Therefore, on–shell the CNM model describes a hyper-Kähler manifold as well. In particular, as also noted in [19], the duality Legendre transform acts on the manifold as a change of coordinates which is in general non–holomorphic (not preserving the complex structures).

6. 6D, $\mathcal{N} = (1,0)$ CNM sigma-models (II): an indirect approach from its dual CC model

In section 4 we have studied the most general CNM sigma–model defined by our ansatz (4.1, 4.2) and worked out the constraints on its defining functions as coming from the direct restoration of 6D Lorentz invariance of the on–shell action. Unfortunately, as already noticed, the system of constraints which we obtain cannot be solved in general and we are not able to easily read from them the geometrical properties of the target space.

In the previous section we have discussed the duality properties between CC and CNM sigma models. This opens the possibility to find the set of constraints satisfied by the CNM sigma–model (4.1, 4.2) by following an alternative, indirect approach: Since we know the precise relation between the geometric tensors of the CNM model and of its dual we can infer the constraints of the CNM case from the hyper-Kähler condition (2.20) for the dual CC model.

Given the general CNM (4.1, 4.2) we can find the components of the two–forms Ω and $\overline{\Omega}$ for the CC dual (5.5)

$$\Omega_{IJ} = Q_{J(I)} - Q_{I(J)} = \begin{pmatrix} \left(P_{b(a)} - P_{a(b)}\right) + \chi_s \left(S^s_{b(a)} - S^s_{a(b)}\right) & -S^r_a \\ S^k_b & 0 \end{pmatrix}, \tag{6.1}$$

$$\overline{\Omega}_{\overline{I}\overline{J}} = \overline{Q}_{\overline{J}(\overline{I})} - \overline{Q}_{\overline{I}(\overline{J})} = \begin{pmatrix} \left(\overline{P}_{\overline{b}(\overline{a})} - \overline{P}_{\overline{a}(\overline{b})}\right) + \overline{\chi}_{\overline{s}} \left(\overline{S}^{\overline{s}}_{\overline{b}(\overline{a})} - \overline{S}^{\overline{s}}_{\overline{a}(\overline{b})}\right) & -\overline{S}^{\overline{r}}_{\overline{a}} \\ \overline{S}^{\overline{k}}_{\overline{b}} & 0 \end{pmatrix}. \tag{6.2}$$

To write the hyper-Kähler conditions (2.20) we need find the expression of the Kähler metric $\widetilde{G}_{I\overline{I}} = \partial_I \partial_{\overline{I}} \widetilde{G}$ of the dual CC geometry in terms of the tensors in the CNM basis. Exploiting the fact that the Kähler potential is the Legendre transform of G, we find

$$\begin{aligned}
\widetilde{G}_{a\overline{a}} &= \partial_{\overline{a}} \left[G_a + G_k \frac{\partial \Sigma^k}{\partial \Phi^a} + G_{\overline{k}} \frac{\partial \overline{\Sigma}^{\overline{k}}}{\partial \Phi^a} - \chi_k \frac{\partial \Sigma^k}{\partial \Phi^a} - \overline{\chi}_{\overline{k}} \frac{\partial \overline{\Sigma}^{\overline{k}}}{\partial \Phi^a} \right] \\
&= G_{a\overline{a}} + G_{ak} \frac{\partial \Sigma^k}{\partial \overline{\Phi}^{\overline{a}}} + G_{a\overline{k}} \frac{\partial \overline{\Sigma}^{\overline{k}}}{\partial \overline{\Phi}^{\overline{a}}} \;=\; G_{a\overline{a}} + G_{\overline{a}k} \frac{\partial \Sigma^k}{\partial \Phi^a} + G_{\overline{a}\overline{k}} \frac{\partial \overline{\Sigma}^{\overline{k}}}{\partial \Phi^a} \quad , \qquad (6.3) \\
\widetilde{G}^k{}_{\overline{a}} &= \partial^k G_{\overline{a}} \;=\; G_{r\overline{a}} \frac{\partial \Sigma^r}{\partial \chi_k} + G_{\overline{a}\overline{r}} \frac{\partial \overline{\Sigma}^{\overline{r}}}{\partial \chi_k} \\
&= \partial_{\overline{a}} \left[G_r \frac{\partial \Sigma^r}{\partial \chi_k} + G_{\overline{r}} \frac{\partial \overline{\Sigma}^{\overline{r}}}{\partial \chi_k} - \frac{\partial \Sigma^r}{\partial \chi_k} \chi_r - \Sigma^k - \frac{\partial \overline{\Sigma}^{\overline{r}}}{\partial \chi_k} \overline{\chi}_{\overline{r}} \right] \;=\; - \frac{\partial \Sigma^k}{\partial \overline{\Phi}^{\overline{a}}} \quad , \qquad (6.4) \\
\widetilde{G}_a{}^{\overline{k}} &= G_{ar} \frac{\partial \Sigma^r}{\partial \overline{\chi}_{\overline{k}}} + G_{a\overline{r}} \frac{\partial \overline{\Sigma}^{\overline{r}}}{\partial \overline{\chi}_{\overline{k}}} \;=\; - \frac{\partial \overline{\Sigma}^{\overline{k}}}{\partial \Phi^a} \quad , \qquad (6.5) \\
\widetilde{G}^{k\overline{k}} &= - \frac{\partial \Sigma^k}{\partial \overline{\chi}_{\overline{k}}} \;=\; - \frac{\partial \overline{\Sigma}^{\overline{k}}}{\partial \chi_k} \quad . \qquad (6.6)
\end{aligned}$$

Using the defining equations (5.3, 5.4) of the CNM Legendre transform we have

$$H \;=\; \begin{pmatrix} G_{kr} & G_{k\overline{r}} \\ G_{\overline{k}r} & G_{\overline{k}\overline{r}} \end{pmatrix} \;=\; \begin{pmatrix} \frac{\partial \chi_k}{\partial \Sigma^r} & \frac{\partial \chi_k}{\partial \overline{\Sigma}^{\overline{r}}} \\ \frac{\partial \overline{\chi}_{\overline{k}}}{\partial \Sigma^r} & \frac{\partial \overline{\chi}_{\overline{k}}}{\partial \overline{\Sigma}^{\overline{r}}} \end{pmatrix} \quad , \qquad (6.7)$$

$$H^{-1} \;\equiv\; \begin{pmatrix} H^{kr} & H^{k\overline{r}} \\ H^{\overline{k}r} & H^{\overline{k}\overline{r}} \end{pmatrix} \;=\; \begin{pmatrix} \frac{\partial \Sigma^k}{\partial \chi_r} & \frac{\partial \Sigma^k}{\partial \overline{\chi}_{\overline{r}}} \\ \frac{\partial \overline{\Sigma}^{\overline{k}}}{\partial \chi_r} & \frac{\partial \overline{\Sigma}^{\overline{k}}}{\partial \overline{\chi}_{\overline{k}}} \end{pmatrix} \quad . \qquad (6.8)$$

We are then able to write the metric of the CC Kähler target space in terms of tensors of the original CNM model

$$\widetilde{G}_{I\overline{J}} = \begin{pmatrix} \widetilde{G}_{a\overline{b}} & \widetilde{G}_a{}^{\overline{r}} \\ \widetilde{G}^k{}_{\overline{b}} & \widetilde{G}^{k\overline{r}} \end{pmatrix} \quad , \qquad (6.9)$$

with

$$\begin{aligned}
\widetilde{G}_{a\overline{a}} &= G_{a\overline{a}} - G_{ak}(H^{kr} G_{r\overline{a}} + H^{k\overline{r}} G_{\overline{r}\overline{a}}) - G_{a\overline{k}}(H^{\overline{k}r} G_{r\overline{a}} + H^{\overline{k}\overline{r}} G_{\overline{r}\overline{a}}) \quad , \qquad (6.10) \\
\widetilde{G}_a{}^{\overline{k}} &= G_{ar} H^{r\overline{k}} + G_{a\overline{r}} H^{\overline{r}\overline{k}} \quad , \qquad (6.11) \\
\widetilde{G}^k{}_{\overline{a}} &= G_{r\overline{a}} H^{rk} + G_{\overline{a}\overline{r}} H^{\overline{r}k} \quad , \qquad (6.12) \\
\widetilde{G}^{k\overline{k}} &= -H^{k\overline{k}} \quad . \qquad (6.13)
\end{aligned}$$

In a matricial form as (4.5, 4.6) and (6.7, 6.8), the dual CC Kähler metric (6.9) can be written as

$$\widetilde{G} \;=\; P_+ \left(M \;-\; N H^{-1} N^T P_- \right) P_2 \;-\; \frac{1}{4} P_2 [P, [P, H^{-1}]] P_- \;+\; \frac{1}{4} [P, [P, N H^{-1}]] \quad , \qquad (6.14)$$

where

$$
P_2 \equiv \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \quad . \tag{6.15}
$$

At this point we impose the hyper-Kähler condition (2.20)

$$
\widetilde{G}_{I\overline{I}} \;=\; -\,\Omega_{IJ}\,\widetilde{G}^{J\overline{J}}\,\overline{\Omega}_{\overline{J}\overline{I}} \quad\Longleftrightarrow\quad \widetilde{G}_{I\overline{I}}\,\Omega^{IJ}\,\widetilde{G}_{J\overline{J}} = -\,\overline{\Omega}_{\overline{I}\overline{J}} \quad . \tag{6.16}
$$

These equations can be re-interpreted as conditions on the CNM tensors, remembering that $\chi_k = G_k(\Phi, \overline{\Phi}, \Sigma, \overline{\Sigma})$. Therefore, inverting the metric $\widetilde{G}_{I\overline{J}}$ or the form Ω_{IJ} one can write explicitly the conditions on the tensors of the original CNM sigma–model which insures on–shell 6D Lorentz and $\mathcal{N} = (1,0)$ SUSY on both sides of the duality map.

We have not investigated extensively the constraints (6.16) for the generic sigma–model (4.1, 4.2) yet. However, we can make few preliminary observations.

So far, we have not specified the number of coordinates of the target space described respectively by chiral Φ^a and nonminimal Σ^k (or dual chiral χ_k) superfields. In order to understand if there are restrictions on the number of coordinates we analyse the tensors Ω_{IJ}, $\overline{\Omega}_{\overline{I}\overline{J}}$ (6.1, 6.2). To have a well–defined dual CC model with on–shell 6D, $\mathcal{N} = (1,0)$ SUSY we know that Ω_{IJ} has to provide a local parametrization of the components of the nondegenerate closed holomorphic two–form of a hyper-Kähler manifold. In particular, the matrix Ω_{IJ} has to be invertible, i.e. its kernel has to be trivial. Observing the explicit expression of Ω_{IJ} (6.1) in the case under consideration it is clear that the number of coordinates described by nonminimal superfields n_{nm} has to be equal or less as the number of chirals n_c ($n_{nm} \leq n_c$). In fact, if $n_{nm} > n_c$ then S_a^k would certainly have a non–trivial kernel and so would Ω_{IJ} (6.1).

We first consider the case $n_{nm} = n_c \equiv n$. Since Ω_{IJ} has to be invertible, we require $S_a^k(\Phi)$ to have a trivial kernel and an inverse $S_k^a(\Phi)$ exists such that $S_b^k S_r^b = \delta_r^k$, $S_r^a S_b^r = \delta_b^a$. If S_a^k is invertible, it is possible to simplify the CC dual (5.5) by doing the holomorphic affine–like χ_k, $\overline{\chi}_{\overline{k}}$ superfield redefinition

$$
\tilde{\chi}_k \;\equiv\; \chi_r\, S_a^r(\Phi)\, \delta_k^a \,+\, P_a(\Phi)\, \delta_k^a \;,\; \overline{\tilde{\chi}}_{\overline{k}} \;\equiv\; \overline{\chi}_{\overline{r}}\, \overline{S}_{\overline{a}}^{\overline{r}}(\overline{\Phi})\, \delta_{\overline{k}}^{\overline{a}} \,+\, \overline{P}_{\overline{a}}(\overline{\Phi})\, \delta_{\overline{k}}^{\overline{a}} \quad , \tag{6.17}
$$

$$
\chi_k \;=\; \tilde{\chi}_r\, \delta_b^r\, S_k^b(\Phi) \,-\, P_a(\Phi)\, \delta_k^a \;,\; \overline{\chi}_{\overline{k}} \;=\; \overline{\tilde{\chi}}_{\overline{r}}\, \delta_{\overline{b}}^{\overline{r}}\, \overline{S}_{\overline{r}}^{\overline{b}}(\overline{\Phi}) \,-\, \overline{P}_{\overline{a}}(\overline{\Phi})\, \delta_{\overline{k}}^{\overline{a}} \quad , \tag{6.18}
$$

keeping the Φ, $\overline{\Phi}$ coordinates fixed. In the $(\Phi, \overline{\Phi}, \tilde{\chi}_k, \overline{\tilde{\chi}}_{\overline{k}})$ target space coordinates the CC sigma–model which we find is in a symplectic basis where the holomorphic term is (5.8). If we now dualize the resulting CC sigma–model with respect to the new tilde coordinates, we find a CNM sigma–model where $P_a \equiv 0$ and $S_a^k(\Phi) \equiv \delta_a^k$. This means that, with $n_{nm} = n_c$ all the consistent 6D, $\mathcal{N} = (1,0)$ CNM sigma–models can be described by $P_a = 0$ and $S_a^k(\Phi) = \delta_a^k$. This is clearly what we expect from the discussion of section 5. We then focus on this particular case.

With $P_a = 0$ and $S_a^k = \delta_a^k$, Ω_{IJ} and Ω^{IJ} become the constant symplectic matrix and its inverse, respectively (also $\Omega_{IJ} = P_2 P$ and $\Omega^{IJ} = -P_2 P = P P_2$). The hyper-Kähler condition on the CC dual sigma–model then reduces to

$$0 = H^{k\overline{k}}\delta^a_k(G_{ap}H^{p\overline{r}} + G_{a\overline{p}}H^{\overline{p}\overline{r}}) - H^{k\overline{r}}\delta^a_k(G_{ap}H^{p\overline{k}} + G_{a\overline{p}}H^{\overline{p}\overline{k}}) \quad , \tag{6.19}$$

$$\begin{aligned} -\delta^{\overline{k}}_{\overline{b}} &= H^{k\overline{k}}\delta^b_k\Big[G_{b\overline{b}} - G_{bs}(H^{sr}G_{r\overline{b}} + H^{s\overline{r}}G_{\overline{r}\overline{b}}) - G_{b\overline{s}}(H^{\overline{s}r}G_{r\overline{b}} + H^{\overline{s}\overline{r}}G_{\overline{r}\overline{b}})\Big] \\ &\quad + (G_{as}H^{s\overline{k}} + G_{a\overline{s}}H^{\overline{s}\overline{k}})\delta^a_k(G_{\overline{b}p}H^{pk} + G_{\overline{b}\overline{p}}H^{\overline{p}k}) \quad , \end{aligned} \tag{6.20}$$

$$\begin{aligned} 0 &= \Bigg[G_{a\overline{a}} - G_{as}(H^{sr}G_{r\overline{a}} + H^{s\overline{r}}G_{\overline{r}\overline{a}}) \\ &\qquad\qquad - G_{a\overline{s}}(H^{\overline{s}r}G_{r\overline{a}} + H^{\overline{s}\overline{r}}G_{\overline{r}\overline{a}})\Bigg]\delta^a_k(G_{\overline{b}p}H^{pk} + G_{\overline{b}\overline{p}}H^{\overline{p}k}) \\ &\quad - \Bigg[G_{a\overline{b}} - G_{as}(H^{sr}G_{r\overline{b}} + H^{s\overline{r}}G_{\overline{r}\overline{b}}) \\ &\qquad\qquad - G_{a\overline{s}}(H^{\overline{s}r}G_{r\overline{b}} + H^{\overline{s}\overline{r}}G_{\overline{r}\overline{b}})\Bigg]\delta^a_k(G_{\overline{a}p}H^{pk} + G_{\overline{a}\overline{p}}H^{\overline{p}k}) \quad , \end{aligned} \tag{6.21}$$

or in the matricial form

$$P_2 P = \widetilde{G}^T\, P_2 P\, \widetilde{G} \quad . \tag{6.22}$$

where $\widetilde{G}$ is given by (6.14).

If $n_{nm} < n_c$ and $n_{nm} + n_c = 2n$, we expect that under a set of partial dualities the CNM model can be mapped to a CNM model with $n_{nm} = n_c$, as discussed in section 5. Therefore, the previous analysis still works. On the other hand, if $n_{nm} + n_c = 2n + 1$ the theory is not well-defined since an odd number of target space coordinates is incompatible with the hyper-Kähler condition.

Example: 4D target space We now analyse the 4D target space example of section 4 (see the action (4.20)) using the indirect approach of this section. The dual CC Kähler metric $\widetilde{G}_{I\overline{J}}$ (6.9) is

$$\widetilde{G}_{\Phi\overline{\Phi}} = G_{\Phi\overline{\Phi}} - \tfrac{1}{\det H}\Big[G_{\Phi\Sigma}(G_{\overline{\Sigma}\overline{\Sigma}}G_{\Sigma\overline{\Phi}} - G_{\Sigma\overline{\Sigma}}G_{\overline{\Phi}\overline{\Sigma}}) + G_{\Phi\overline{\Sigma}}(G_{\Sigma\Sigma}G_{\overline{\Phi}\overline{\Sigma}} - G_{\Sigma\overline{\Sigma}}G_{\Sigma\overline{\Phi}})\Big] \ , \tag{6.23}$$

$$\widetilde{G}_{\Phi}{}^{\overline{\Sigma}} = \tfrac{1}{\det H}\Big[G_{\Phi\overline{\Sigma}}G_{\Sigma\Sigma} - G_{\Phi\Sigma}G_{\Sigma\overline{\Sigma}}\Big] \ , \tag{6.24}$$

$$\widetilde{G}^{\Sigma}{}_{\overline{\Phi}} = \tfrac{1}{\det H}\Big[G_{\Sigma\overline{\Phi}}G_{\overline{\Sigma}\overline{\Sigma}} - G_{\overline{\Phi}\overline{\Sigma}}G_{\Sigma\overline{\Sigma}}\Big] \ , \tag{6.25}$$

$$\widetilde{G}^{\Sigma\overline{\Sigma}} = \tfrac{1}{\det H}\, G_{\Sigma\overline{\Sigma}} \quad . \tag{6.26}$$

Imposing the hyper-Kähler condition (6.16) with $\Omega_{IJ} = \Omega^0_{IJ} = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$, the only constraint on $G(\Phi, \overline{\Phi}, \Sigma, \overline{\Sigma})$ which arises is (4.28). This proves that, at least in the case of a four dimensional target space geometry, our direct and indirect approaches are equivalent. Furthermore, from the present discussion it follows that (4.28) is effectively a hyper-Kähler condition as we claimed at the end of section 4.

7. Conclusions

In this work, we have endeavored to open a discussion of 6D supersymmetric nonlinear sigma-models. Our specific techniques involved utilizing 4D superfields, thus keeping manifest this degree of supersymmetry, that permit full 6D Lorentz invariance to be realized only on-shell.

We have demonstrated, as might have been expected, that the use of two chiral superfields to represent the 6D $\mathcal{N} = (1, 0)$ hypermultiplet provides the simplest manner in which to describe such actions. This formulation has the interesting feature that to write its action requires in addition to a Kähler potential, a holomorphic U(1)-bundle connection. The 6D Lorentz invariance imposes a condition that relates the Kähler potential to the connection in such a way that the sigma-model manifold must be hyper-Kähler. The field strength of the holomorphic U(1)-bundle connection has been found to be related to the well-known triplet of covariantly constant complex structures. We have also given a brief introduction to the use of projective superspace for analysis of this class of models. As the polar multiplets of projective superspace necessarily lead to combinations of chiral and nonmininal multiplets (CNM's) making their appearance, we finally have studied this class of models by an analysis based directly on the introduction of CNM actions without the use of projective superspace. In this last set of activities, general conditions were derived, but owing to the sheer algebraic complexity, we have shown that there exist well define special cases which demonstrate the equivalence of the CNM description, where possible.

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A. Some definitions and formulae

In analogy to the four dimensional case [16 − 18] we define the component fields of the CNM multiplet (4.2) as

$$
\begin{aligned}
A^a &= \Phi^a| \quad , \quad \psi^a_\alpha = D_\alpha \Phi^a| \quad , \quad F^a = D^2\Phi^a| \quad , \\
\overline{A}^{\overline{a}} &= \overline{\Phi}^{\overline{a}}| \quad , \quad \overline{\psi}^{\overline{a}}_{\dot\alpha} = \overline{D}_{\dot\alpha}\overline{\Phi}^{\overline{a}}| \quad , \quad \overline{F}^{\overline{a}} = \overline{D}^2\overline{\Phi}^{\overline{a}}| \quad , \\
B^k &= \Sigma^k| \quad , \quad \overline{\zeta}^k_{\dot\alpha} = \overline{D}_{\dot\alpha}\Sigma^k| \quad , \quad H^k = D^2\Sigma^k| \quad , \\
\rho^k_\alpha &= D_\alpha\Sigma^k| \quad , \quad p^k_{\alpha\dot\alpha} = \overline{D}_{\dot\alpha} D_\alpha \Sigma^k| \quad , \quad \overline{\beta}^k_{\dot\alpha} = \tfrac{1}{2} D^\alpha \overline{D}_{\dot\alpha} D_\alpha \Sigma^k| \quad . \\
\overline{B}^{\overline{k}} &= \overline{\Sigma}^{\overline{k}}| \quad , \quad \zeta^{\overline{k}}_\alpha = D_\alpha \overline{\Sigma}^{\overline{k}}| \quad , \quad \overline{H}^{\overline{k}} = \overline{D}^2\overline{\Sigma}^{\overline{k}}| \quad , \\
\overline{\rho}^{\overline{k}}_{\dot\alpha} &= \overline{D}_{\dot\alpha}\overline{\Sigma}^{\overline{k}}| \quad , \quad \overline{p}^{\overline{k}}_{\alpha\dot\alpha} = -D_\alpha \overline{D}_{\dot\alpha}\overline{\Sigma}^{\overline{k}}| \quad , \quad \beta^{\overline{k}}_\alpha = \tfrac{1}{2}\overline{D}^{\dot\alpha} D_\alpha \overline{D}_{\dot\alpha}\overline{\Sigma}^{\overline{k}}| \quad .
\end{aligned}
\tag{A.1}
$$

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From the bosonic components we define the vectors

$$\mathcal{A} \equiv \begin{pmatrix} A^a \\ \overline{A}^{\overline{a}} \end{pmatrix} \quad , \quad \mathcal{B} = \begin{pmatrix} B^k \\ \overline{B}^{\overline{k}} \end{pmatrix} \quad , \tag{A.2}$$

$$\mathcal{F} \equiv \begin{pmatrix} F^a \\ \overline{F}^{\overline{a}} \end{pmatrix} \quad , \quad \mathcal{H} \equiv \begin{pmatrix} H^k \\ \overline{H}^{\overline{k}} \end{pmatrix} \quad , \quad \mathcal{P}_{\alpha\dot{\alpha}} \equiv \begin{pmatrix} p^k_{\alpha\dot{\alpha}} \\ \overline{p}^{\overline{k}}_{\alpha\dot{\alpha}} \end{pmatrix} \quad . \tag{A.3}$$

B. 6D CNM sigma–model action in components

Now we give the expression of the action in components for the general CNM sigma–model described by (4.1) with constraints (4.2). By non-trivial dimensional reduction we can obtain component actions in lower dimensions. In particular, the sigma–model actions we obtain can contain non–trivial mass and potential terms coming from the CNM constraint. These actions generalize sigma–models studied in [16, 17] where only the standard non-minimal constraint $\overline{D}^2\Sigma = 0$ was considered. Our more general models are relevant for a CNM description of SUSY theories with non-trivial central charges.

With the components defined as (A.1), the action of the CNM sigma–model (4.1, 4.2) is

$$\begin{aligned}
&\Big(P_{b(a)} - P_{a(b)}\Big)\Big(F^a\partial A^b + \frac{1}{2}\psi^{b\alpha}\partial\,\psi^a_\alpha\Big) + \frac{1}{2}\Big(P_{b(ac)} - P_{a(bc)}\Big)(\partial A^a)\,\psi^{b\alpha}\psi^c_\alpha \\
&+\Big(\overline{P}_{\overline{b}(\overline{a})} - \overline{P}_{\overline{a}(\overline{b})}\Big)\Big(\overline{F}^{\overline{a}}\overline{\partial}\,\overline{A}^{\overline{b}} + \frac{1}{2}\overline{\psi}^{\overline{b}\dot{\alpha}}\overline{\partial}\,\overline{\psi}^{\overline{a}}_{\dot{\alpha}}\Big) + \frac{1}{2}\Big(\overline{P}_{\overline{b}\,(\overline{a}\overline{c})} - \overline{P}_{\overline{a}\,(\overline{b}\overline{c})}\Big)(\overline{\partial}\,\overline{A}^{\overline{a}})\,\overline{\psi}^{\overline{b}\dot{\alpha}}\overline{\psi}^{\overline{c}}_{\dot{\alpha}} \\
&+G_k S^k_{a(b)}F^b\partial\,A^a + G_k S^k_a\partial F^a + G_k S^k_{a(b)}\psi^{b\alpha}\partial\,\psi^a_\alpha + G_k S^k_{a(bc)}(\partial A^a)\psi^{b\alpha}\psi^c_\alpha \\
&+\overline{G}_{\overline{k}}\overline{S}^{\overline{k}}_{\overline{a}(\overline{b})}\overline{F}^{\overline{b}}\overline{\partial}\,\overline{A}^{\overline{a}} + \overline{G}_{\overline{k}}\overline{S}^{\overline{k}}_{\overline{a}}\overline{\partial}\,\overline{F}^{\overline{a}} + \overline{G}_{\overline{k}}\overline{S}^{\overline{k}}_{\overline{a}(\overline{b})}\overline{\psi}^{\overline{b}\dot{\alpha}}\overline{\partial}\,\overline{\psi}^{\overline{a}}_{\dot{\alpha}} + \overline{G}_{\overline{k}}\overline{S}^{\overline{k}}_{\overline{a}(\overline{b}\overline{c})}(\overline{\partial}\,\overline{A}^{\overline{a}})\overline{\psi}^{\overline{b}\dot{\alpha}}\overline{\psi}^{\overline{c}}_{\dot{\alpha}} \\
&+G_{ak}\Big[\,S^k_b F^a\partial\,A^b + S^k_b\psi^{a\alpha}\partial\,\psi^b_\alpha + S^k_{b(c)}(\partial\,A^b)\psi^{a\alpha}\psi^c_\alpha\,\Big] \\
&+G_{\overline{a}\overline{k}}\Big[\,\overline{S}^{\overline{k}}_{\overline{b}}\overline{F}^{\overline{a}}\overline{\partial}\,\overline{A}^{\overline{b}} + \overline{S}^{\overline{k}}_{\overline{b}}\overline{\psi}^{\overline{a}\dot{\alpha}}\overline{\partial}\,\overline{\psi}^{\overline{b}}_{\dot{\alpha}} + \overline{S}^{\overline{k}}_{\overline{b}(\overline{c})}(\overline{\partial}\,\overline{A}^{\overline{b}})\overline{\psi}^{\overline{a}\dot{\alpha}}\overline{\psi}^{\overline{c}}_{\dot{\alpha}}\,\Big] \\
&+G_{kr}\Big[\,S^k_a H^r\partial\,A^a + \frac{1}{2}\partial^{\alpha\dot{\alpha}}B^k\partial_{\alpha\dot{\alpha}}B^r + p^{k\alpha\dot{\alpha}}i\partial_{\alpha\dot{\alpha}}B^r - \frac{1}{2}p^{k\alpha\dot{\alpha}}p^r_{\alpha\dot{\alpha}} - \overline{\zeta}^{k\dot{\alpha}}\overline{\beta}^r_{\dot{\alpha}} \\
&\qquad + \rho^{r\alpha}\Big(\frac{1}{2}i\partial_{\alpha\dot{\alpha}}\overline{\zeta}^{k\dot{\alpha}} + S^k_a\partial\,\psi^a_\alpha + S^k_{a(b)}(\partial\,A^a)\psi^b_\alpha\Big)\Big] \\
&+G_{\overline{k}\overline{r}}\Big[\,\overline{S}^{\overline{k}}_{\overline{a}}\overline{H}^{\overline{r}}\overline{\partial}\,\overline{A}^{\overline{a}} + \frac{1}{2}\partial^{\alpha\dot{\alpha}}\overline{B}^{\overline{k}}\partial_{\alpha\dot{\alpha}}\overline{B}^{\overline{r}} - \overline{p}^{\overline{k}\alpha\dot{\alpha}}i\partial_{\alpha\dot{\alpha}}\overline{B}^{\overline{r}} - \frac{1}{2}\overline{p}^{\overline{k}\alpha\dot{\alpha}}\overline{p}^{\overline{r}}_{\alpha\dot{\alpha}} - \zeta^{\overline{k}\alpha}\beta^{\overline{r}}_\alpha \\
&\qquad + \overline{\rho}^{\overline{r}\dot{\alpha}}\Big(\frac{1}{2}i\partial_{\alpha\dot{\alpha}}\zeta^{\overline{k}\alpha} + \overline{S}^{\overline{k}}_{\overline{a}}\overline{\partial}\,\overline{\psi}^{\overline{a}}_{\dot{\alpha}} + \overline{S}^{\overline{k}}_{\overline{a}(\overline{b})}(\overline{\partial}\,\overline{A}^{\overline{a}})\overline{\psi}^{\overline{b}}_{\dot{\alpha}}\Big)\Big] \\
&+G_{a\overline{a}}\Big[-\frac{1}{2}\partial^{\alpha\dot{\alpha}}\overline{A}^{\overline{a}}\partial_{\alpha\dot{\alpha}}A^a - \frac{1}{2}\overline{\psi}^{\overline{a}}_{\dot{\alpha}}i\partial^{\alpha\dot{\alpha}}\psi^a_\alpha - \frac{1}{2}\psi^a_\alpha i\partial^{\alpha\dot{\alpha}}\overline{\psi}^{\overline{a}}_{\dot{\alpha}} + \overline{F}^{\overline{a}}F^a\Big]
\end{aligned}$$

$$
\begin{aligned}
&+ G_{k\overline{k}}\Bigg[\frac{1}{2}\partial^{\alpha\dot\alpha}\overline{B}^{\overline{k}}\partial_{\alpha\dot\alpha}B^k + \frac{1}{2}\overline{\zeta}^{k}_{\dot\alpha} i\partial^{\alpha\dot\alpha}\zeta^{\overline{k}}_{\alpha} + \frac{1}{2}\zeta^{\overline{k}}_{\alpha} i\partial^{\alpha\dot\alpha}\overline{\zeta}^{k}_{\dot\alpha} - \overline{p}^{\overline{k}\alpha\dot\alpha}p^k_{\alpha\dot\alpha} + S^k_a\overline{S}^{\overline{k}}_{\overline{a}}(\overline{\partial}\,\overline{A}^{\overline{a}})(\partial\, A^a) \\
&\qquad + \overline{H}^{\overline{k}}H^k + S^k_a\zeta^{\overline{k}\alpha}\partial\,\psi^a_\alpha + S^k_{a(b)}(\partial\, A^a)\zeta^{\overline{k}\alpha}\psi^b_\alpha - \beta^{\overline{k}\alpha}\rho^k_\alpha + \overline{S}^{\overline{k}}_{\overline{a}}\overline{\zeta}^{k\dot\alpha}\overline{\partial}\,\overline{\psi}^{\overline{a}}_{\dot\alpha} \\
&\qquad + \overline{S}^{\overline{k}}_{\overline{a}(\overline{b})}(\overline{\partial}\,\overline{A}^{\overline{a}})\overline{\zeta}^{k\dot\alpha}\overline{\psi}^{\overline{b}}_{\dot\alpha} - \overline{\beta}^{k\dot\alpha}\overline{\rho}^{\overline{k}}_{\dot\alpha}\Bigg] \\
&+ G_{a\overline{k}}\Bigg[\frac{1}{2}\partial^{\alpha\dot\alpha}\overline{B}^{\overline{k}}\partial_{\alpha\dot\alpha}A^a - \overline{p}^{\overline{k}\alpha\dot\alpha}i\partial_{\alpha\dot\alpha}A^a + \overline{H}^{\overline{k}}F^a - \psi^{a\alpha}\beta^{\overline{k}}_\alpha - \frac{1}{2}\overline{\rho}^{\overline{k}}_{\dot\alpha}i\partial^{\alpha\dot\alpha}\psi^a_\alpha\Bigg] \\
&+ G_{k\overline{a}}\Bigg[\frac{1}{2}\partial^{\alpha\dot\alpha}\overline{A}^{\overline{a}}\partial_{\alpha\dot\alpha}B^k + p^{k\alpha\dot\alpha}i\partial_{\alpha\dot\alpha}\overline{A}^{\overline{a}} + \overline{F}^{\overline{a}}H^k - \overline{\psi}^{\overline{a}\dot\alpha}\overline{\beta}^k_{\dot\alpha} - \frac{1}{2}\rho^k_\alpha i\partial^{\alpha\dot\alpha}\overline{\psi}^{\overline{a}}_{\dot\alpha}\Bigg] \\
&+ G_{ak\overline{k}}\Bigg[\frac{i}{2}(\partial^{\alpha\dot\alpha}A^a)\zeta^{\overline{k}}_\alpha\overline{\zeta}^k_{\dot\alpha} + \frac{i}{2}(\partial^{\alpha\dot\alpha}A^a)\rho^k_\alpha\overline{\rho}^{\overline{k}}_{\dot\alpha} + F^a\overline{\zeta}^{k\dot\alpha}\overline{\rho}^{\overline{k}}_{\dot\alpha} + \overline{H}^{\overline{k}}\psi^{a\alpha}\rho^k_\alpha + S^k_b(\partial\, A^b)\psi^{a\alpha}\zeta^{\overline{k}}_\alpha \\
&\qquad + \overline{p}^{\overline{k}\alpha\dot\alpha}\psi^a_\alpha\overline{\zeta}^k_{\dot\alpha} - \frac{i}{2}(\partial^{\alpha\dot\alpha}B^k)\psi^a_\alpha\overline{\rho}^{\overline{k}}_{\dot\alpha} + p^{k\alpha\dot\alpha}\psi^a_\alpha\overline{\rho}^{\overline{k}}_{\dot\alpha}\Bigg] \\
&+ G_{k\overline{a}\overline{k}}\Bigg[\frac{i}{2}(\partial^{\alpha\dot\alpha}\overline{A}^{\overline{a}})\overline{\zeta}^k_{\dot\alpha}\zeta^{\overline{k}}_\alpha + \frac{i}{2}(\partial^{\alpha\dot\alpha}\overline{A}^{\overline{a}})\overline{\rho}^{\overline{k}}_{\dot\alpha}\rho^k_\alpha + \overline{F}^{\overline{a}}\zeta^{\overline{k}\alpha}\rho^k_\alpha + H^k\overline{\psi}^{\overline{a}\dot\alpha}\overline{\rho}^{\overline{k}}_{\dot\alpha} + \overline{S}^{\overline{k}}_{\overline{b}}(\overline{\partial}\,\overline{A}^{\overline{b}})\overline{\psi}^{\overline{a}\dot\alpha}\overline{\zeta}^k_{\dot\alpha} \\
&\qquad - p^{k\alpha\dot\alpha}\overline{\psi}^{\overline{a}}_{\dot\alpha}\zeta^{\overline{k}}_\alpha - \frac{i}{2}(\partial^{\alpha\dot\alpha}\overline{B}^{\overline{k}})\overline{\psi}^{\overline{a}}_{\dot\alpha}\rho^k_\alpha - \overline{p}^{\overline{k}\alpha\dot\alpha}\overline{\psi}^{\overline{a}}_{\dot\alpha}\rho^k_\alpha\Bigg] \\
&+ G_{kr\overline{k}}\Bigg[\frac{i}{2}(\partial^{\alpha\dot\alpha}B^r)\overline{\zeta}^k_{\dot\alpha}\zeta^{\overline{k}}_\alpha + \frac{i}{2}(\partial^{\alpha\dot\alpha}B^r)\overline{\rho}^{\overline{k}}_{\dot\alpha}\rho^k_\alpha + \frac{1}{2}\overline{S}^{\overline{k}}_{\overline{a}}(\overline{\partial A}^{\overline{a}})\overline{\zeta}^{k\dot\alpha}\overline{\zeta}^r_{\dot\alpha} + p^{r\alpha\dot\alpha}\rho^k_\alpha\overline{\rho}^{\overline{k}}_{\dot\alpha} + \overline{p}^{\overline{k}\alpha\dot\alpha}\rho^k_\alpha\overline{\zeta}^r_{\dot\alpha} \\
&\qquad + p^{r\alpha\dot\alpha}\zeta^{\overline{k}}_\alpha\overline{\zeta}^k_{\dot\alpha} + \frac{i}{2}(\partial^{\alpha\dot\alpha}\overline{B}^{\overline{k}})\rho^k_\alpha\overline{\zeta}^r_{\dot\alpha} + S^k_a(\partial\, A^a)\rho^{r\alpha}\zeta^{\overline{k}}_\alpha + H^r\overline{\rho}^{\overline{k}\dot\alpha}\overline{\zeta}^k_{\dot\alpha} + \frac{1}{2}\overline{H}^{\overline{k}}\rho^{k\alpha}\rho^r_\alpha\Bigg] \\
&+ G_{k\overline{k}\overline{r}}\Bigg[\frac{i}{2}(\partial^{\alpha\dot\alpha}\overline{B}^{\overline{r}})\zeta^{\overline{k}}_\alpha\overline{\zeta}^k_{\dot\alpha} + \frac{i}{2}(\partial^{\alpha\dot\alpha}\overline{B}^{\overline{r}})\rho^k_\alpha\overline{\rho}^{\overline{k}}_{\dot\alpha} + \frac{1}{2}S^k_a(\partial A^a)\zeta^{\overline{k}\alpha}\zeta^{\overline{r}}_\alpha + \overline{p}^{\overline{r}\alpha\dot\alpha}\rho^k_\alpha\overline{\rho}^{\overline{k}}_{\dot\alpha} + p^{k\alpha\dot\alpha}\zeta^{\overline{r}}_\alpha\overline{\rho}^{\overline{k}}_{\dot\alpha} \\
&\qquad + \overline{p}^{\overline{r}\alpha\dot\alpha}\zeta^{\overline{k}}_\alpha\overline{\zeta}^k_{\dot\alpha} + \frac{i}{2}(\partial^{\alpha\dot\alpha}B^k)\overline{\rho}^{\overline{k}}_{\dot\alpha}\zeta^{\overline{r}}_\alpha + \overline{S}^{\overline{k}}_{\overline{a}}(\overline{\partial}\,\overline{A}^{\overline{a}})\overline{\rho}^{\overline{r}\dot\alpha}\overline{\zeta}^k_{\dot\alpha} + \overline{H}^{\overline{r}}\rho^{k\alpha}\zeta^{\overline{k}}_\alpha + \frac{1}{2}H^k\overline{\rho}^{\overline{k}\dot\alpha}\overline{\rho}^{\overline{r}}_{\dot\alpha}\Bigg] \\
&+ G_{akr}\Bigg[(i\partial^{\alpha\dot\alpha}B^r)\overline{\zeta}^k_{\dot\alpha}\psi^a_\alpha + \frac{i}{2}(\partial^{\alpha\dot\alpha}A^a)\rho^k_\alpha\overline{\zeta}^r_{\dot\alpha} + S^k_b(\partial A^b)\psi^{a\alpha}\rho^r_\alpha + \frac{1}{2}F^a\overline{\zeta}^{k\dot\alpha}\overline{\zeta}^r_{\dot\alpha} + p^{r\alpha\dot\alpha}\psi^a_\alpha\overline{\zeta}^k_{\dot\alpha}\Bigg] \\
&+ G_{\overline{a}\overline{k}\overline{r}}\Bigg[(i\partial^{\alpha\dot\alpha}\overline{B}^{\overline{r}})\zeta^{\overline{k}}_\alpha\overline{\psi}^{\overline{a}}_{\dot\alpha} + \frac{i}{2}(\partial^{\alpha\dot\alpha}\overline{A}^{\overline{a}})\overline{\rho}^{\overline{k}}_{\dot\alpha}\zeta^{\overline{r}}_\alpha + \overline{S}^{\overline{k}}_{\overline{b}}(\overline{\partial}\,\overline{A}^{\overline{b}})\overline{\psi}^{\overline{a}\dot\alpha}\overline{\rho}^{\overline{r}}_{\dot\alpha} + \frac{1}{2}\overline{F}^{\overline{a}}\zeta^{\overline{k}\alpha}\zeta^{\overline{r}}_\alpha + \overline{p}^{\overline{r}\alpha\dot\alpha}\zeta^{\overline{k}}_\alpha\overline{\psi}^{\overline{a}}_{\dot\alpha}\Bigg] \\
&+ G_{krs}\Bigg[\frac{1}{2}H^k\overline{\zeta}^{r\dot\alpha}\overline{\zeta}^s_{\dot\alpha} + \frac{i}{2}(\partial^{\alpha\dot\alpha}B^k)\overline{\zeta}^r_{\dot\alpha}\rho^s_\alpha + p^{k\alpha\dot\alpha}\rho^r_\alpha\overline{\zeta}^s_{\dot\alpha} + \frac{1}{2}S^k_a(\partial A^a)\rho^{r\alpha}\rho^s_\alpha\Bigg] \\
&+ G_{\overline{k}\overline{r}\overline{s}}\Bigg[\frac{1}{2}\overline{H}^{\overline{k}}\zeta^{\overline{r}\alpha}\zeta^{\overline{s}}_\alpha + \frac{i}{2}(\partial^{\alpha\dot\alpha}\overline{B}^{\overline{k}})\zeta^{\overline{r}}_\alpha\overline{\rho}^{\overline{s}}_{\dot\alpha} + \overline{p}^{\overline{k}\alpha\dot\alpha}\zeta^{\overline{r}}_\alpha\overline{\rho}^{\overline{s}}_{\dot\alpha} + \frac{1}{2}\overline{S}^{\overline{k}}_{\overline{a}}(\overline{\partial}\,\overline{A}^{\overline{a}})\overline{\rho}^{\overline{r}\dot\alpha}\overline{\rho}^{\overline{s}}_{\dot\alpha}\Bigg] \\
&+ \frac{1}{2}G_{abk}S^k_c(\partial A^c)\psi^{a\alpha}\psi^b_\alpha \;+\; \frac{1}{2}G_{\overline{a}\overline{b}\overline{k}}\overline{S}^{\overline{k}}_{\overline{c}}(\overline{\partial}\,\overline{A}^{\overline{c}})\overline{\psi}^{\overline{a}\dot\alpha}\overline{\psi}^{\overline{b}}_{\dot\alpha}
\end{aligned}
$$

$$
\begin{aligned}
&+ G_{ab\overline{a}}\left[\frac{i}{2}(\partial^{\alpha\dot\alpha}A^{a})\psi^{b}_{\alpha}\overline{\psi}^{\overline{a}}_{\dot\alpha} + \frac{1}{2}\overline{F}^{\overline{a}}\psi^{a\alpha}\psi^{b}_{\alpha}\right] + G_{a\overline{a}\overline{b}}\left[\frac{i}{2}(\partial^{\alpha\dot\alpha}\overline{A}^{\overline{a}})\overline{\psi}^{\overline{b}}_{\dot\alpha}\psi^{a}_{\alpha} + \frac{1}{2}F^{a}\overline{\psi}^{\overline{a}\dot\alpha}\overline{\psi}^{\overline{b}}_{\dot\alpha}\right] \\
&+ G_{ak\overline{a}}\left[\frac{i}{2}(\partial^{\alpha\dot\alpha}B^{k})\overline{\psi}^{\overline{a}}_{\dot\alpha}\psi^{a}_{\alpha} + (i\partial^{\alpha\dot\alpha}\overline{A}^{\overline{a}})\overline{\zeta}^{k}_{\dot\alpha}\psi^{a}_{\alpha} + \frac{i}{2}(\partial^{\alpha\dot\alpha}A^{a})\rho^{k}_{\alpha}\overline{\psi}^{\overline{a}}_{\dot\alpha} + F^{a}\overline{\zeta}^{k\dot\alpha}\overline{\psi}^{\overline{a}}_{\dot\alpha}\right. \\
&\qquad\left. +\ \overline{F}^{\overline{a}}\psi^{a\alpha}\rho^{k}_{\alpha} + p^{k\alpha\dot\alpha}\psi^{a}_{\alpha}\overline{\psi}^{\overline{a}}_{\dot\alpha}\right] \\
&+ G_{a\overline{a}\overline{k}}\left[\frac{i}{2}(\partial^{\alpha\dot\alpha}\overline{B}^{\overline{k}})\psi^{a}_{\alpha}\overline{\psi}^{\overline{a}}_{\dot\alpha} + (i\partial^{\alpha\dot\alpha}A^{a})\zeta^{\overline{k}}_{\alpha}\overline{\psi}^{\overline{a}}_{\dot\alpha} + \frac{i}{2}(\partial^{\alpha\dot\alpha}\overline{A}^{\overline{a}})\overline{\rho}^{\overline{k}}_{\dot\alpha}\psi^{a}_{\alpha} + \overline{F}^{\overline{a}}\zeta^{\overline{k}\alpha}\psi^{a}_{\alpha}\right. \\
&\qquad\left. +\ F^{a}\overline{\psi}^{\overline{a}\dot\alpha}\overline{\rho}^{\overline{k}}_{\dot\alpha} + \overline{p}^{\overline{k}\alpha\dot\alpha}\psi^{a}_{\alpha}\overline{\psi}^{\overline{a}}_{\dot\alpha}\right] \\
&+ G_{ab\overline{k}}\left[\frac{1}{2}\overline{H}^{\overline{k}}\psi^{a\alpha}\psi^{b}_{\alpha} + \frac{1}{2}(\partial^{\alpha\dot\alpha}A^{b})\psi^{a}_{\alpha}\overline{\rho}^{\overline{k}}_{\dot\alpha}\right] + G_{k\overline{a}\overline{b}}\left[\frac{1}{2}H^{k}\overline{\psi}^{\overline{a}\dot\alpha}\overline{\psi}^{\overline{b}}_{\dot\alpha} + \frac{i}{2}(\partial^{\alpha\dot\alpha}\overline{A}^{\overline{b}})\overline{\psi}^{\overline{a}}_{\dot\alpha}\rho^{k}_{\alpha}\right] \\
&+ G_{kr\overline{a}}\left[H^{r}\overline{\psi}^{\overline{a}\dot\alpha}\overline{\zeta}^{k}_{\dot\alpha} + \frac{i}{2}(\partial^{\alpha\dot\alpha}\overline{A}^{\overline{a}})\overline{\zeta}^{k}_{\dot\alpha}\rho^{r}_{\alpha} + \frac{i}{2}(\partial^{\alpha\dot\alpha}B^{r})\overline{\psi}^{\overline{a}}_{\dot\alpha}\rho^{k}_{\alpha} + p^{r\alpha\dot\alpha}\rho^{k}_{\alpha}\overline{\psi}^{\overline{a}}_{\dot\alpha} + \frac{1}{2}\overline{F}^{\overline{a}}\rho^{k\alpha}\rho^{r}_{\alpha}\right] \\
&+ G_{a\overline{k}\overline{r}}\left[\overline{H}^{\overline{r}}\psi^{a\alpha}\zeta^{\overline{k}}_{\alpha} + \frac{i}{2}(\partial^{\alpha\dot\alpha}A^{a})\zeta^{\overline{k}}_{\alpha}\overline{\rho}^{\overline{r}}_{\dot\alpha} + \frac{i}{2}(\partial^{\alpha\dot\alpha}\overline{B}^{\overline{r}})\psi^{a}_{\alpha}\overline{\rho}^{\overline{k}}_{\dot\alpha} + \overline{p}^{\overline{r}\alpha\dot\alpha}\psi^{a}_{\alpha}\overline{\rho}^{\overline{k}}_{\dot\alpha} + \frac{1}{2}F^{a}\overline{\rho}^{\overline{k}\dot\alpha}\overline{\rho}^{\overline{r}}_{\dot\alpha}\right] \\
&+ \frac{1}{4}G_{ab\overline{a}\overline{b}}\,\psi^{a\alpha}\psi^{b}_{\alpha}\,\overline{\psi}^{\overline{a}\dot\alpha}\overline{\psi}^{\overline{b}}_{\dot\alpha} \ +\ \frac{1}{2}G_{ak\overline{a}\overline{b}}\,\psi^{a\alpha}\rho^{k}_{\alpha}\,\overline{\psi}^{\overline{a}\dot\alpha}\overline{\psi}^{\overline{b}}_{\dot\alpha} \ +\ \frac{1}{2}G_{ab\overline{a}\overline{k}}\,\psi^{a\alpha}\psi^{b}_{\alpha}\,\overline{\psi}^{\overline{a}\dot\alpha}\overline{\rho}^{\overline{k}}_{\dot\alpha} \\
&+ G_{ak\overline{a}\overline{k}}\left(\psi^{a\alpha}\zeta^{\overline{k}}_{\alpha}\,\overline{\zeta}^{k\dot\alpha}\overline{\psi}^{\overline{a}}_{\dot\alpha} + \psi^{a\alpha}\rho^{k}_{\alpha}\,\overline{\psi}^{\overline{a}\dot\alpha}\overline{\rho}^{\overline{k}}_{\dot\alpha}\right) \ +\ \frac{1}{2}G_{abk\overline{a}}\,\psi^{a\alpha}\psi^{b}_{\alpha}\,\overline{\psi}^{\overline{a}\dot\alpha}\overline{\zeta}^{k}_{\dot\alpha} \\
&+ \frac{1}{2}G_{a\overline{a}\overline{b}\overline{k}}\,\psi^{a\alpha}\zeta^{\overline{k}}_{\alpha}\,\overline{\psi}^{\overline{a}\dot\alpha}\overline{\psi}^{\overline{b}}_{\dot\alpha} \ +\ G_{a\overline{a}\overline{k}\overline{r}}\,\psi^{a\alpha}\zeta^{\overline{k}}_{\alpha}\,\overline{\psi}^{\overline{a}\dot\alpha}\overline{\rho}^{\overline{r}}_{\dot\alpha} \ +\ G_{akr\overline{a}}\,\psi^{a\alpha}\rho^{k}_{\alpha}\,\overline{\zeta}^{r\dot\alpha}\overline{\psi}^{\overline{a}}_{\dot\alpha} \\
&+ \frac{1}{4}G_{abkr}\,\psi^{a\alpha}\psi^{b}_{\alpha}\,\overline{\zeta}^{k\dot\alpha}\overline{\zeta}^{r}_{\dot\alpha} \ +\ \frac{1}{4}G_{\overline{a}\overline{b}\overline{k}\overline{r}}\,\zeta^{\overline{k}\alpha}\zeta^{\overline{r}}_{\alpha}\,\overline{\psi}^{\overline{a}\dot\alpha}\overline{\psi}^{\overline{b}}_{\dot\alpha} \ +\ \frac{1}{2}G_{akrs}\,\psi^{a\alpha}\rho^{k}_{\alpha}\,\overline{\zeta}^{r\dot\alpha}\overline{\zeta}^{s}_{\dot\alpha} \\
&+ \frac{1}{2}G_{\overline{a}\overline{k}\overline{r}\overline{s}}\,\zeta^{\overline{k}\alpha}\zeta^{\overline{r}}_{\alpha}\,\overline{\psi}^{\overline{a}\dot\alpha}\overline{\rho}^{\overline{s}}_{\dot\alpha} \ +\frac{1}{4}G_{krst}\,\rho^{k\alpha}\rho^{r}_{\alpha}\,\overline{\zeta}^{s\dot\alpha}\overline{\zeta}^{t}_{\dot\alpha} \ +\frac{1}{4}G_{\overline{k}\overline{r}\overline{s}\overline{t}}\,\zeta^{\overline{k}\alpha}\zeta^{\overline{r}}_{\alpha}\,\overline{\rho}^{\overline{s}\dot\alpha}\overline{\rho}^{\overline{t}}_{\dot\alpha} \\
&+ G_{akr\overline{k}}\left(\psi^{a\alpha}\rho^{k}_{\alpha}\,\overline{\zeta}^{r\dot\alpha}\overline{\rho}^{\overline{k}}_{\dot\alpha} + \frac{1}{2}\psi^{a\alpha}\zeta^{\overline{k}}_{\alpha}\,\overline{\zeta}^{k\dot\alpha}\overline{\zeta}^{r}_{\dot\alpha}\right) \ +\ \frac{1}{2}G_{abk\overline{k}}\,\psi^{a\alpha}\psi^{b}_{\alpha}\,\overline{\zeta}^{k\dot\alpha}\overline{\rho}^{\overline{k}}_{\dot\alpha} \\
&+ \frac{1}{2}G_{k\overline{a}\overline{b}\overline{k}}\,\rho^{k\alpha}\zeta^{\overline{k}}_{\alpha}\,\overline{\psi}^{\overline{a}\dot\alpha}\overline{\psi}^{\overline{b}}_{\dot\alpha} \ +\ G_{k\overline{a}\overline{k}\overline{r}}\left(\rho^{k\alpha}\zeta^{\overline{k}}_{\alpha}\,\overline{\psi}^{\overline{a}\dot\alpha}\overline{\rho}^{\overline{r}}_{\dot\alpha} + \frac{1}{2}\zeta^{\overline{k}\alpha}\zeta^{\overline{r}}_{\alpha}\,\overline{\psi}^{\overline{a}\dot\alpha}\overline{\zeta}^{k}_{\dot\alpha}\right) \\
&+ G_{ak\overline{k}\overline{r}}\left(\psi^{a\alpha}\zeta^{\overline{k}}_{\alpha}\,\overline{\zeta}^{k\dot\alpha}\overline{\rho}^{\overline{r}}_{\dot\alpha} + \frac{1}{2}\psi^{a\alpha}\rho^{k}_{\alpha}\,\overline{\rho}^{\overline{k}\dot\alpha}\overline{\rho}^{\overline{r}}_{\dot\alpha}\right) \ +\ \frac{1}{4}G_{ab\overline{k}\overline{r}}\,\psi^{a\alpha}\psi^{b}_{\alpha}\,\overline{\rho}^{\overline{k}\dot\alpha}\overline{\rho}^{\overline{r}}_{\dot\alpha} \\
&+ \frac{1}{4}G_{kr\overline{a}\overline{b}}\,\rho^{k\alpha}\rho^{r}_{\alpha}\,\overline{\psi}^{\overline{a}\dot\alpha}\overline{\psi}^{\overline{b}}_{\dot\alpha} \ +\ G_{kr\overline{a}\overline{k}}\left(\rho^{k\alpha}\zeta^{\overline{k}}_{\alpha}\,\overline{\zeta}^{r\dot\alpha}\overline{\psi}^{\overline{a}}_{\dot\alpha} + \frac{1}{2}\rho^{k\alpha}\rho^{r}_{\alpha}\,\overline{\psi}^{\overline{a}\dot\alpha}\overline{\rho}^{\overline{k}}_{\dot\alpha}\right) \\
&+ \frac{1}{2}G_{krs\overline{k}}\left(\rho^{k\alpha}\zeta^{\overline{k}}_{\alpha}\,\overline{\zeta}^{r\dot\alpha}\overline{\zeta}^{s}_{\dot\alpha} + \rho^{k\alpha}\rho^{r}_{\alpha}\,\overline{\zeta}^{s\dot\alpha}\overline{\rho}^{\overline{k}}_{\dot\alpha}\right) \ +\ \frac{1}{2}G_{krs\overline{a}}\,\rho^{k\alpha}\rho^{r}_{\alpha}\,\overline{\zeta}^{s\dot\alpha}\overline{\psi}^{\overline{a}}_{\dot\alpha}
\end{aligned}
$$

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$$
+\frac{1}{2}G_{a\overline{k}\overline{r}\overline{s}}\,\psi^{a\alpha}\zeta_{\alpha}^{\overline{k}}\,\overline{\rho}^{\overline{r}\dot{\alpha}}\overline{\rho}_{\dot{\alpha}}^{\overline{s}} \;+\; \frac{1}{2}G_{k\overline{k}\overline{r}\overline{s}}\left(\zeta^{\overline{k}\alpha}\zeta_{\alpha}^{\overline{r}}\,\overline{\zeta}^{k\dot{\alpha}}\overline{\rho}_{\dot{\alpha}}^{\overline{s}}+\rho^{k\alpha}\zeta_{\alpha}^{\overline{k}}\,\overline{\rho}^{\overline{r}\dot{\alpha}}\overline{\rho}_{\dot{\alpha}}^{\overline{s}}\right)
$$

$$
+G_{kr\overline{k}\overline{r}}\left(\rho^{k\alpha}\zeta_{\alpha}^{\overline{k}}\,\overline{\zeta}^{r\dot{\alpha}}\overline{\rho}_{\dot{\alpha}}^{\overline{r}}+\frac{1}{4}\rho^{k\alpha}\rho_{\alpha}^{r}\,\overline{\rho}^{\overline{k}\dot{\alpha}}\overline{\rho}_{\dot{\alpha}}^{\overline{r}}+\frac{1}{4}\zeta^{\overline{k}\alpha}\zeta_{\alpha}^{\overline{r}}\,\overline{\zeta}^{k\dot{\alpha}}\overline{\zeta}_{\dot{\alpha}}^{r}\right) \quad . \tag{B.1}
$$

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